Exploratory Experimentation and Computation

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CARMA

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"[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.

"In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting."

George Polya (**1887-**1985)

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

Revised 03/12/12

ICERM Workshop on

Reproducibility in Computational and Experimental Mathematics

December 10 to 14, 2012

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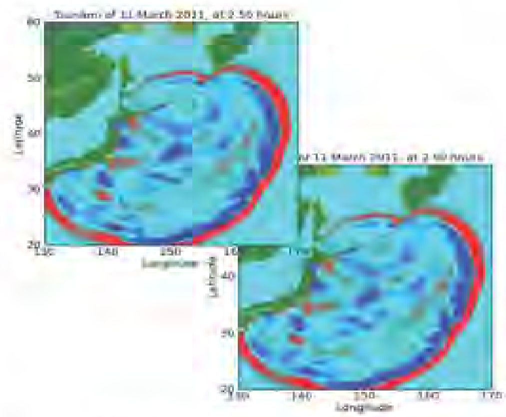
Reproducibility in Computational and Experimental Mathematics (December 10-14, 2012)

Description



Click for code to create this image.

In addition to advancing research and discovery in pure and applied mathematics, computation is pervasive across the sciences and now computational research results are more crucial than ever for public policy, risk management, and national security. Reproducibility of carefully documented experiments is a cornerstone of the scientific method, and yet is often lacking in computational mathematics, science, and engineering. Setting and achieving appropriate standards for reproducibility in computation poses a number of interesting technological and social challenges. The purpose of this workshop is to discuss aspects of reproducibility most relevant to the mathematical sciences among researchers from pure and applied mathematics from academics and other settings, together with interested parties from funding agencies, national laboratories, professional societies, and publishers. This will be a working workshop, with relatively few talks and dedicated time for breakout group discussions on the current state of the art and the tools, policies, and infrastructure that are needed to improve the situation. The groups will be charged with developing guides to current best practices and/or white papers on desirable advances.

Speaker and Participant List

CLICK HERE TO PARTICIPATE

Organizing Committee

David H. Bailey
(Lawrence Berkeley National Laboratory)

Jon Borwein
(Centre for Computer Assisted Research Mathematics and its Applications)

Randall J. LeVeque
(University of Washington)

Bill Rider
(Sandia National Laboratory)

William Stein
(University of Washington)

Victoria Stodden
(Columbia University)

ICERM Workshop on

Reproducibility in Computational and Experimental Mathematics

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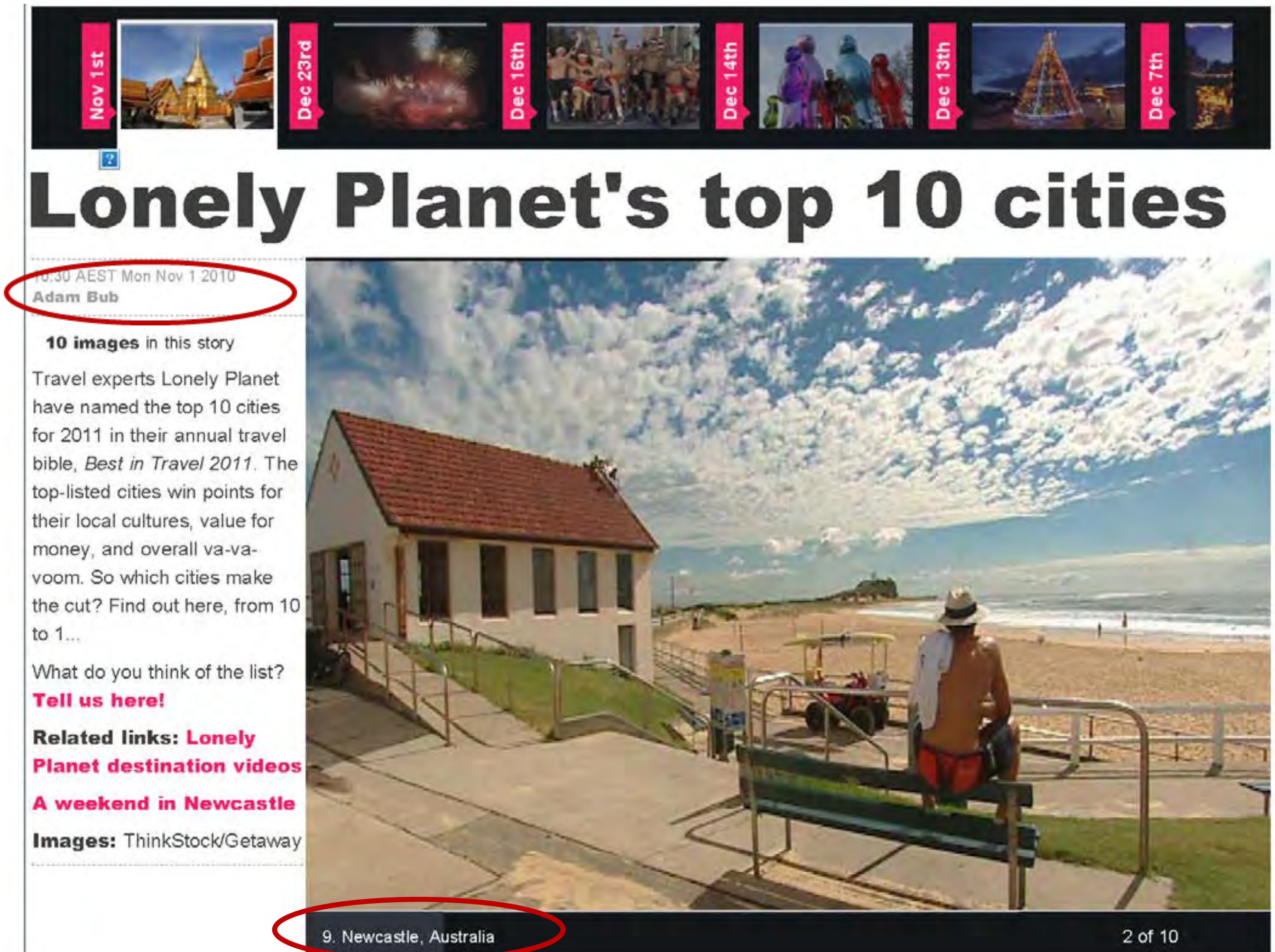


Nov 1st Dec 23rd Dec 16th Dec 14th Dec 13th Dec 7th

Lonely Planet's top 10 cities

10:30 AEST Mon Nov 1 2010
Adam Bub

10 images in this story

Travel experts Lonely Planet have named the top 10 cities for 2011 in their annual travel bible, *Best in Travel 2011*. The top-listed cities win points for their local cultures, value for money, and overall va-va-voom. So which cities make the cut? Find out here, from 10 to 1...

What do you think of the list? **Tell us here!**

Related links: Lonely Planet destination videos

A weekend in Newcastle

Images: ThinkStock/Getaway

9. Newcastle, Australia   2 of 10

Where I now live and work

(red)wine

home

Come visit CARMA

ABSTRACT

Jonathan M. Borwein
Newcastle

Abstract: The mathematical research community is facing a great challenge to re-evaluate the role of proof in light of the growing power of current computer systems, of modern mathematical computing packages, and of the growing capacity to data-mine on the Internet. Add to that the enormous complexity of many modern capstone results such as the Poincaré conjecture, Fermat's last theorem, and the Classification of finite simple groups. As the need and prospects for inductive mathematics blossom, the requirement to ensure the role of proof is properly founded remains undiminished. I shall look at the philosophical context with examples and then offer some of five bench-marking examples of the opportunities and challenges we face. (Related paper with DHB, NAMS, November 2011)

"***The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it*.**" – Jacques Hadamard (1865-1963)

OUTLINE

I. **Working Definitions and Examples of**:

- Discovery
- Proof (and of Mathematics)
- Digital-Assistance
- Experimentation (in Maths and in Science)
- Reproducibility and Simplification

II. (**Some few of**) Five Numbers:

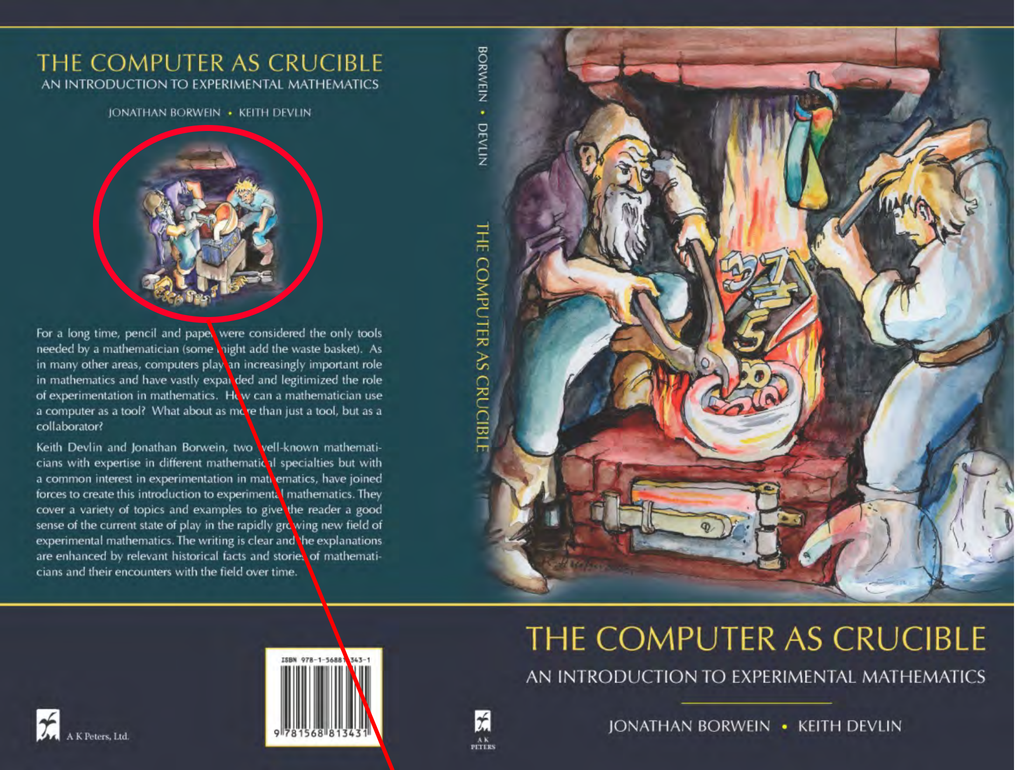
- p(n)
- Pi
- tau(n)
- zeta(3)
- 1/Pi

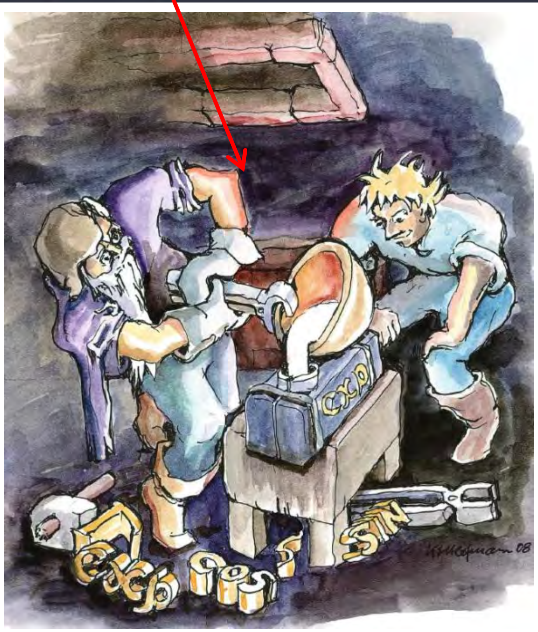
"*Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it*."
- Sir Alec Cairncross, 1996

III. A Cautionary Finale

IV. Making Some Tacit Conclusions Explicit

"*Mathematical proofs like diamonds should be hard and clear, and will be touched with nothing but strict reasoning*." - John Locke





Jonathan Borwein

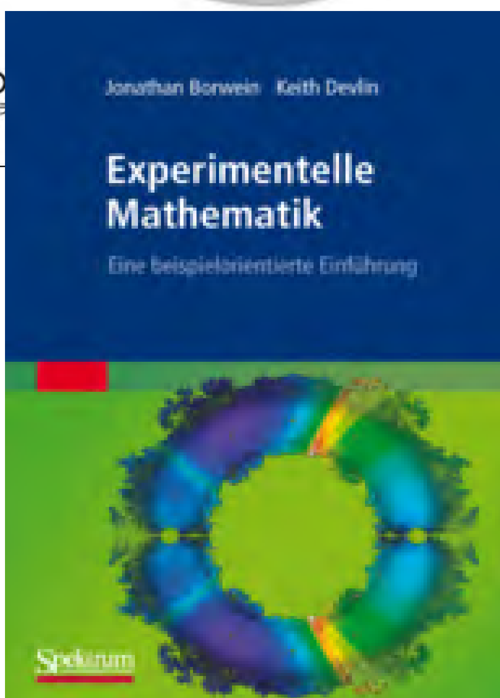
Keith Devlin

with illustrations by Karl H. Hofmann

Contents

AK Peters 2008 Japan & Germany 2010

Cookbook Mathematics



- ✓State of the art machine translation
  - ✓"math magic melting pot"
  - ✓"full head mathematicians"
- ✓No wonder Sergei Brin wants more

PART I. PHILOSOPHY, PSYCHOLOGY, ETC

*“ **This is the essence of science. Even though I do not understand quantum mechanics or the nerve cell membrane, I trust those who do. Most scientists are quite ignorant about most sciences but all use a shared grammar that allows them to recognize their craft when they see it.***

The motto of the Royal Society of London is **'Nullius in verba'** *: trust not in words. Observation and experiment are what count, not opinion and introspection. Few working scientists have much respect for those who try to interpret nature in metaphysical terms.* ***For most wearers of white coats, philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it.*** *Outside of psychology it plays almost no part in the functions of the research machine.”* - Steve Jones

- From his 1997 NYT BR review of Steve Pinker’s *How the Mind Works*.

WHAT is a DISCOVERY?

"discovering a truth has three components. First, there is the independence requirement, which is just that one comes to believe the proposition concerned by one's own lights, without reading it or being told. Secondly, there is the requirement that one comes to believe it in a reliable way. Finally, there is the requirement that one's coming to believe it involves no violation of one's epistemic state. … **In short , discovering a truth is coming to believe it in an independent, reliable, and rational way**."

Marcus Giaquinto, Visual Thinking in Mathematics. An Epistemological Study, p. 50, OUP 2007

"***All truths are easy to understand once they are discovered; the point is to discover them.***" – Galileo Galilei

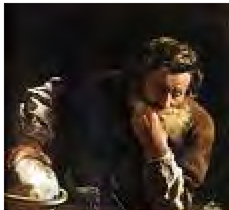
Galileo was not alone in this view

*"I will send you the proofs of the theorems in this book. Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. **I am convinced that this is no less useful for finding proofs of these same theorems.***

*For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. **For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge**."* - Archimedes (287-212 BCE)

Archimedes to Eratosthenes in the introduction to *The Method* in

Mario Livio's, *Is God a Mathematician*? Simon and Schuster, 2009



1a. A Recent Discovery (July 2009 - 2012)

("independent, reliable and rational")

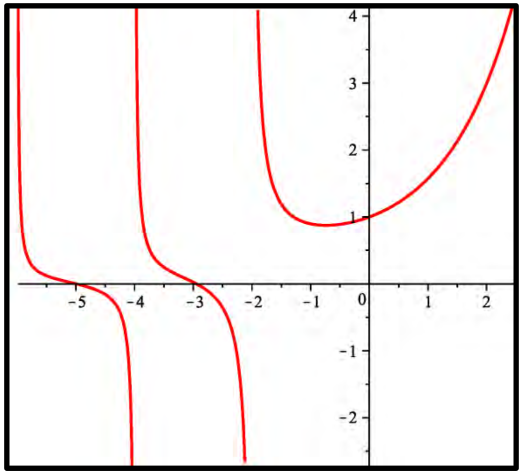
The n-dimensional integral

$$W_n(s) := \int_0^1 \int_0^1 \cdots \int_0^1 \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s dx_1\, dx_2 \cdots dx_n$$

occurs in the study of uniform random walks in the plane.

$W_n(1)$ is the expected distance moved after n steps.

W3(s)



$$W_1(1) = 1 \qquad W_2(1) = \frac{4}{\pi}$$

Pearson (1906)

$$W_3(1) \overset{?}{=} \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right). \quad (1)$$

1000 3 step walks

(1) was checked to 175 places on 256 cores in about 15 minutes. It orginated with our discover (later proof JMB-Nuyens-Straub-Wan) that for $k = 0, 1, 2, 3, \ldots$

$$W_3(2k) = {}_3F_2\left({\frac{1}{2}, -k, -k \atop 1, 1}\Big|4\right) \text{ and } W_3(1) \overset{?}{=} \mathrm{Re}\,{}_3F_2\left({\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \atop 1, 1}\Big|4\right)$$

We proved the formula for $2k$ (counts abelian squares) and *numerically* saw it was half-true at $k = 1/2$

WHAT is MATHEMATICS?

MATHEMATICS, n. a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.

- This definition, from my *Collins* Dictionary has no mention of proof, nor the means of reasoning to be allowed (vidé Giaquinto). *Webster's* contrasts:

INDUCTION, n. any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.

and

DEDUCTION, n. **a.** a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.

b. a conclusion reached by this process.

"*If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics*." - Kurt Gödel (in his 1951 Gibbs Lecture) echoes of Quine

WHAT is a PROOF?

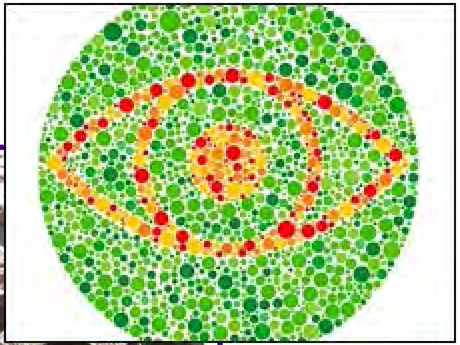
"**PROOF,** *n.* a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the *conclusion* , is the statement of which the truth is thereby established. A *direct proof* proceeds linearly from premises to conclusion; an *indirect proof* (also called reductio ad absurdum) assumes the falsehood of the desired conclusion and shows that to be impossible. See also induction, deduction, valid."

Borowski & JB, Collins Dictionary of Mathematics

INDUCTION, n. 3. (Logic) *a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them.* ***Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.***

"***No. I have been teaching it all my life, and I do not want to have my ideas upset***." - Isaac Todhunter (1820-1884) recording Maxwell's response when asked whether he would like to see an experimental demonstration of conical refraction.

Decide for yourself



WHAT is DIGITAL ASSISTANCE?

- **Use of Modern Mathematical Computer Packages**
  - Symbolic, Numeric, Geometric, Graphical, …
- **Use of More Specialist Packages or General Purpose Languages**
  - Fortran, C++, **CPLEX**, GAP, PARI, MAGMA, Cinderella …
- **Use of Web Applications**
  - Sloane's Encyclopedia, Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java, Weeks' Topological Games, Polymath (Sci. Amer.), …
- **Use of Web Databases**
  - Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, DLMF, MacTutor, Amazon, …, Kindle Reader, Wolfram Alpha (??)
- All entail **data-mining** ["exploratory experimentation" and "widening technology" as in pharmacology, astrophysics, biotech, … (Franklin)]
  - Clearly the boundaries are blurred and getting blurrier
  - Judgments of a given source's quality vary and are context dependent

"***Knowing things is very 20th century. You just need to be able to find things***."- Danny Hillis on how Google has already **changed how we think** in Achenblog, July 1 2008

- changing cognitive styles

Exploratory Experimentation

Franklin argues that Steinle's "exploratory experimentation" facilitated by "widening technology", as in pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a "local model" is not now prerequisite.

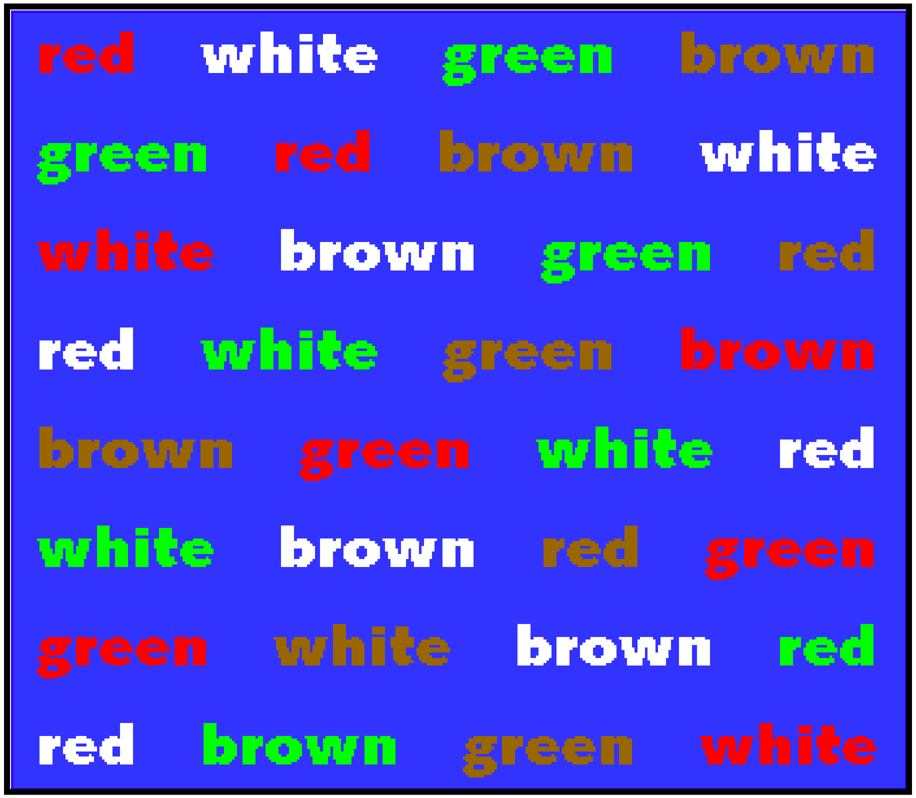
Hendrik Sørenson (2011) cogently makes the case that experimental mathematics (as 'defined' below) is following similar tracks:

> *"These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation is also pertinent to mathematics."*

In consequence, boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting more so.

Changing User Experience and Expectations

What is attention? (**Stroop** test, 1935)



1. Say the **color** represented by the word.
2. Say the **color** represented by the **font** color.

High (young) multitaskers perform #2 very easily. They are great at suppressing information.

http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

Acknowledgements: Cliff Nass, CHIME lab, Stanford (interference and twitter?)

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

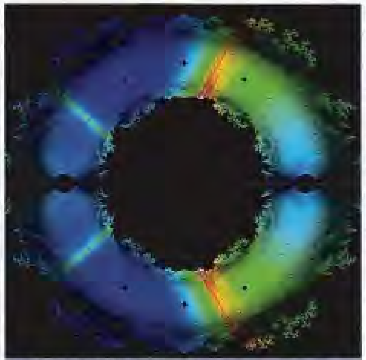
Science News 2004

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results



UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

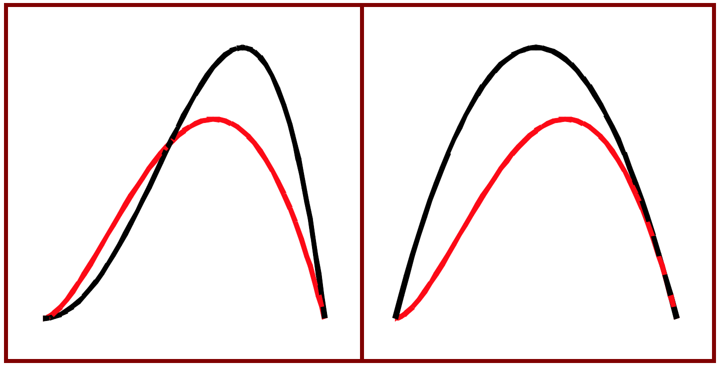
For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-



Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

Reproducibility and Simplification

A recent result (BCC) is that all "box integrals" for integer n, and dimensions $1, 2, 3, 4, 5$ are *hyperclosed*. Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)}\, dx\, dy. \qquad (1)$$

"The computer is claiming its intelligence is real, and ours is artificial."

For instance, *Mathematica* helped us obtain a **100,000** character "closed form" for (1). When $t = 2$, I hand-simplified this to

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right),$$

Here $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1} \sin(n\theta)/n^2$ (simplest non-elementary Fourier series).

Reproducibility and Simplification

"7x8=52, 9-3=5... things like that."

- Automating such reductions will require a sophisticated simplification scheme with a very large and extensible knowledge base.
- With Research Assistant, Alex Kaiser, we have started to design PSLQ-based software to refine and automate this process, `http://www.carma.newcastle.edu.au/jon/auto.pdf`.
- Also semi-automated integrity checking becomes pressing when—as for $J(2)$ or $J(3)$—verifiable output from a symbolic manipulation can be the length of a Salinger novella (10^5 characters or more).
- We now have code that does quite well at that: finding 20 errors in 200 formulae and autocorrecting 17.

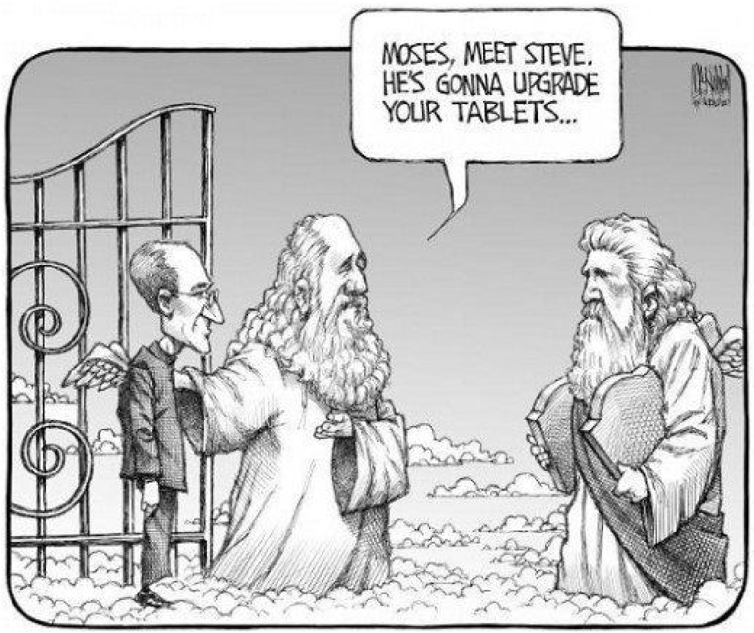
See JMB and REC, "Closed Forms: what they are", *Notices*, Jan 2013.

A Teraflop on a MacPro

"As of early **2011** one will be able to order an Apple desktop machine with appropriate graphics (GPU) cards and software, to achieve on certain problems a teraflop.

Moreover, double-precision floats on GPUs will finally be available. So, again on certain problems, this will be 1000**x** or so faster than we desk-denizens are. REC"

2012: 17 hex digits of pi at 10^{15} position computed by Ed Karrel at NVIDIA (too hard for Blue Gene)



1. What is that number? (1995-2009)

In **I995** or so Andrew Granville emailed me the number

$$\alpha := 1.433127426722312\ldots$$

and challenged me to identify it (our inverse calculator was new in those days).

Changing representations, I asked for its continued fraction? It was

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots] \qquad (1)$$

I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_0(2)}{I_1(2)}$$

where I_0 and I_1 are Bessel functions of the first kind. (Actually I knew that all arithmetic continued fractions arise in such fashion).

In 2010 there are at least **three** other strategies:

- Given (1), type "arithmetic progression", "continued fraction" into Google
- Type "1,4,3,3,1,2,7,4,2" into Sloane's Encyclopaedia of Integer Sequences

I illustrate the outcomes on the next few slides:

“arithmetic progression”, “continued fraction”

In Google on October 15 2008 the first three hits were

Continued Fraction **Constant -- from Wolfram MathWorld**

- 3 visits - 14/09/07Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. **...**
*mathworld.wolfram.com/***ContinuedFraction***Constant.html - 31k*

HAKMEM -- ***CONTINUED FRACTIONS*** **-- DRAFT, NOT YET PROOFED**

The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is I (2/D) A/D [A+D, A+2D, A+3D, . **...**
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

On simple ***continued fractions*** **with partial quotients in** ***arithmetic*** **...**

0. This means that the sequence of partial quotients of the *continued fractions* under. investigation consists of finitely many *arithmetic progressions* (with **...**
www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the MathWorld entry includes

$$[A + D, A + 2D, A + 3D, \ldots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$.

In the Integer Sequence Data Base

AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,4,3,3,1,2,7,4,2

Search

Hints

Search: 1,4,3,3,1,2,7,4,2

Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A060997 Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ... +20

5

1, 4, 3, 3, 1, 2, 7, 4, 2, 6, 7, 2, 2, 3, 1, 1, 7, 5, 8, 3, 1, 7, 1, 8, 3, 4, 5, 5, 7, 7, 5, 9, 9, 1, 8, 2, 0, 4, 3, 1, 5, 1, 2, 7, 6, 7, 9, 0, 5, 9, 8, 0, 5, 2, 3, 4, 3, 4, 4, 2, 8, 6, 3, 6, 3, 9, 4, 3, 0, 9, 1, 8, 3, 2, 5, 4, 1, 7, 2, 9, 0, 0, 1, 3, 6, 5, 0, 3, 7, 2, 6, 4, 3, 5, 7, 8, 6, 1, 1, 4, 6, 5, 9, 5, 0 (list; cons; graph; listen)

OFFSET 1,2

COMMENT The value of this continued fraction is the ratio of two Bessel functions: BesselI(0,2)/BesselI(1,2) = A070910/A096789. Or, equivalently, to the ratio of the sums: sum_{n=0..inf} 1/(n!n!) and sum_{n=0..inf} n/(n!n!). - Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003

FORMULA 1/A052119.

EXAMPLE C=1.433127426722311758317183455775 ...

MATHEMATICA RealDigits[FromContinuedFraction[Range[44]], 10, 110] [[1]]
(* Or *) RealDigits[BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]]
(* Or *) RealDigits[Sum[1/(n!n!), {n, 0, Infinity}] / Sum[n/(n!n!), {n, 0, Infinity}], 10, 110] [[1]]

CROSSREFS Cf. A052119, A001053.
Adjacent sequences: A060994 A060995 A060996 this_sequence A060998 A060999 A061000
Sequence in context: A016699 A060373 A090280 this_sequence A129624 A019975 A073871

KEYWORD cons,easy,nonn

AUTHOR Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001

The **Inverse Calculator** returns

Best guess:

BesI(0,2)/BesI(1,2)

- We show the ISC on another number next
- Most functionality of ISC is built into "identify" in Maple.
- There's also Wolfram ®

"***The price of metaphor is eternal vigilance***." - Arturo Rosenblueth & Norbert Wiener quoted by R. C. Leowontin, *Science* p.1264, Feb 16, 2001 [Human Genome Issue].

Drive NSERC CRSNG Maplesoft

The ISC in Action

The Inverse Symbolic Calculator (ISC) uses a combination of lookup tables and integer relation algorithms in order to associate with a user-defined, truncated decimal expansion (represented as a floating point expression) a closed form representation for the real number.

ISC+ inverse symbolic calculator

Standard lookup results for **12.587886229548403854**

exp(1)+Pi^2

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

3.146264370 Try it!

19.99909998 Try it!

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

The ISC presently accepts either floating point expressions or correct Maple syntax as input. However, for Maple syntax requiring too long for evaluation, a timeout has been implemented.

Visit

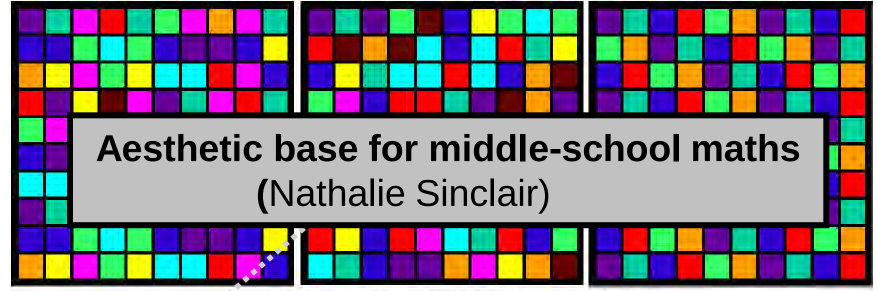
Jon Borwein's Webpage

David Bailey's Webpage

Math Resources Portal

- ISC+ now runs at CARMA
- Less lookup & more algorithms than 1995

1b. A Colour and an Inverse Calculator (1995 & 2007)

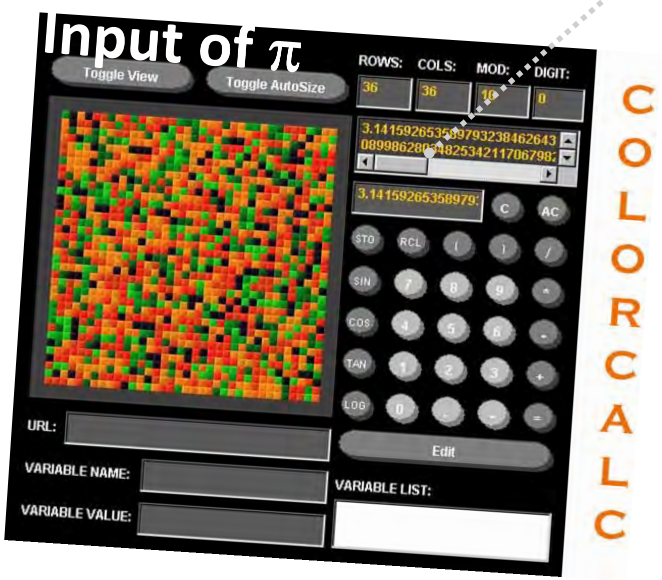


Archimedes: 223/71 < π < 22/7

Inverse Symbolic Computation

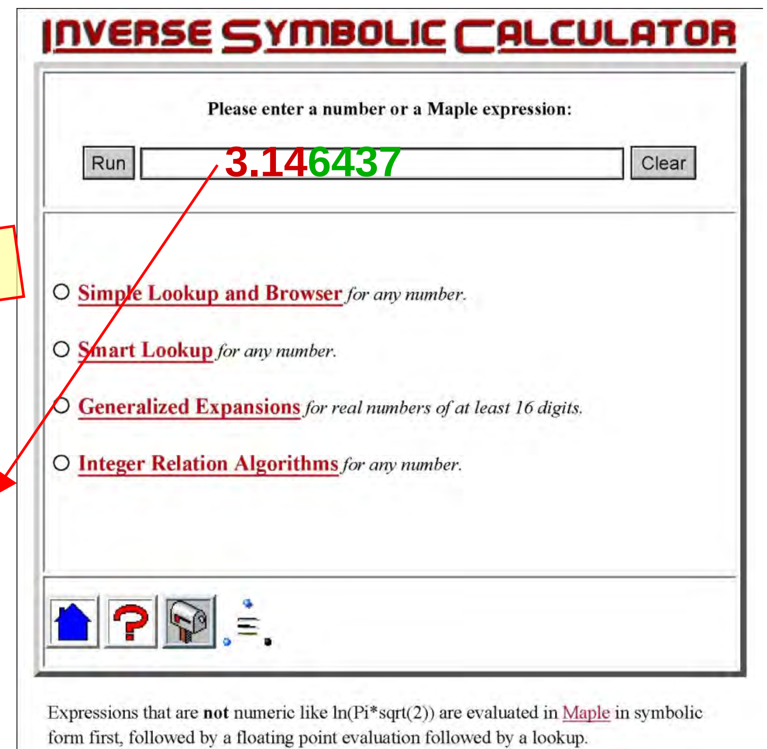
Inferring mathematical structure from numerical data

- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs ***micro-parallelism.*** *In Python since 2007.*
- It faces the "curse of exponentiality"
- Implemented as **identify** since Maple 9.5

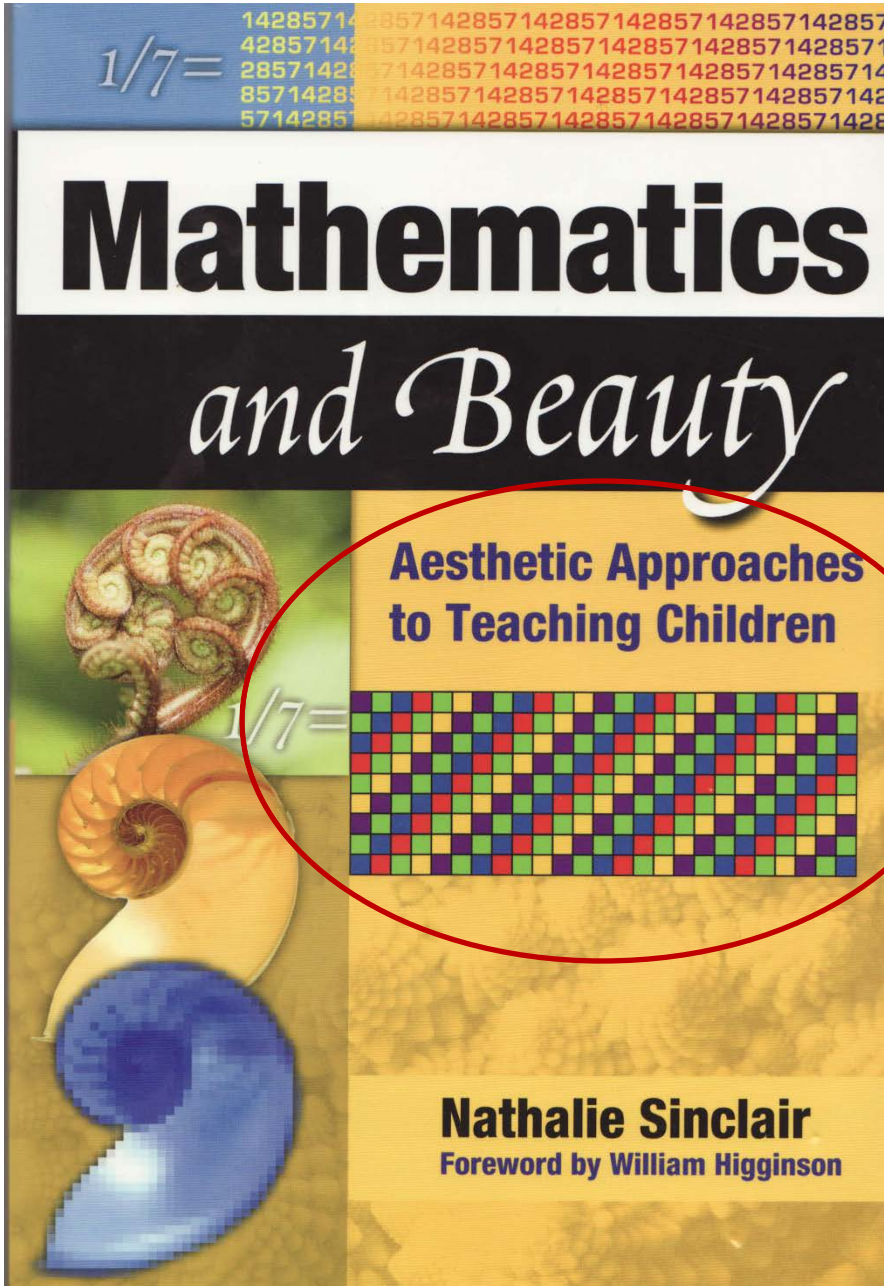


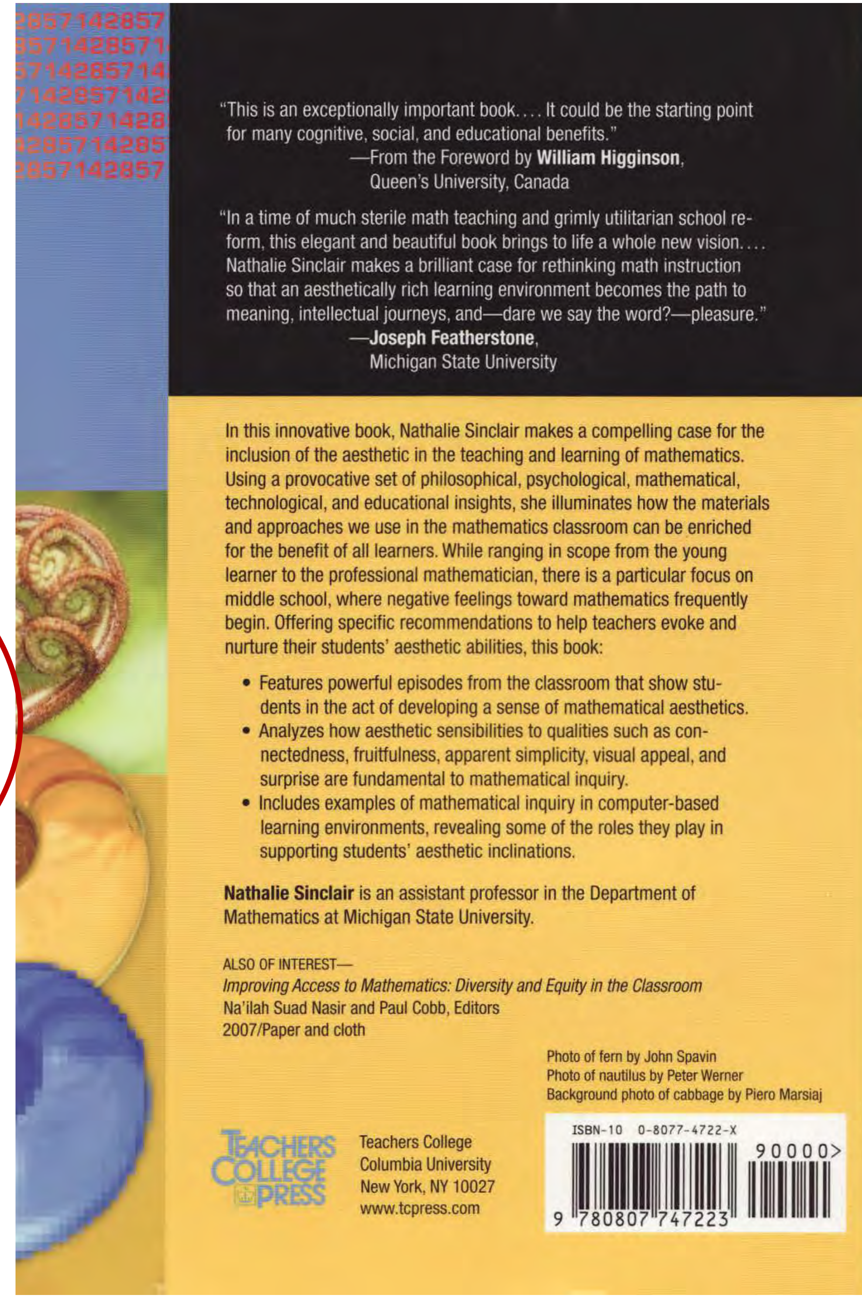
identify(sqrt(2.)+sqrt(3.))

$$\sqrt{2}+\sqrt{3}$$



Mathematics and Beauty (2006)





1c. Exploring Combinatorial Matrices (1993-2008)

In the course of studying **multiple** zeta values we needed to obtain the closed form partial fraction decomposition for

$$\frac{1}{x^s(1-x)^t} = \sum_{j\geq 0} \frac{a_j^{s,t}}{x^j} + \sum_{j\geq 0} \frac{b_j^{s,t}}{(1-x)^j} \qquad a_j^{s,t} = \binom{s+t-j-1}{s-j}$$

This was known to Euler but is easily discovered in Maple.

We needed also to show that **M=A+B-C** is **invertible** where the n by n matrices A, B, C respectively had entries

$$(-1)^{k+1}\binom{2n-j}{2n-k}, \quad (-1)^{k+1}\binom{2n-j}{k-1}, \quad (-1)^{k+1}\binom{j-1}{k-1}$$

Thus, A and C are triangular and B is full.

After messing with many cases I thought to ask for M's minimal polynomial

```
> linalg[minpoly](M(12),t);
```
$-2+t+t^2$

```
> linalg[minpoly](B(20),t);
```
$-1+t^3$

```
> linalg[minpoly](A(20),t);
```
$-1+t^2$

```
> linalg[minpoly](C(20),t);
```
$-1+t^2$

$$M(6) = \begin{bmatrix} 1 & -22 & 110 & -330 & 660 & -924 \\ 0 & -10 & 55 & -165 & 330 & -462 \\ 0 & -7 & 36 & -93 & 162 & -210 \\ 0 & -5 & 25 & -56 & 78 & -84 \\ 0 & -3 & 15 & -31 & 35 & -28 \\ 0 & -1 & 5 & -10 & 10 & -6 \end{bmatrix}$$

The Matrices Conquered

Once this was discovered proving that for all n >2

$$A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2$$

is a nice combinatorial exercise (by hand or computer). Clearly then

$$B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I$$

and the formula

$$M^{-1} = \frac{M + I}{2}$$

is again a fun exercise in formal algebra; as is confirming that we have discovered an amusing presentation of the symmetric group S_3.

- characteristic and minimal polynomials --- which were rather abstract for me as a student --- now become members of a rapidly growing box of symbolic tools, as do many matrix decompositions, etc …

- a typical matrix has a full degree minimal polynomial

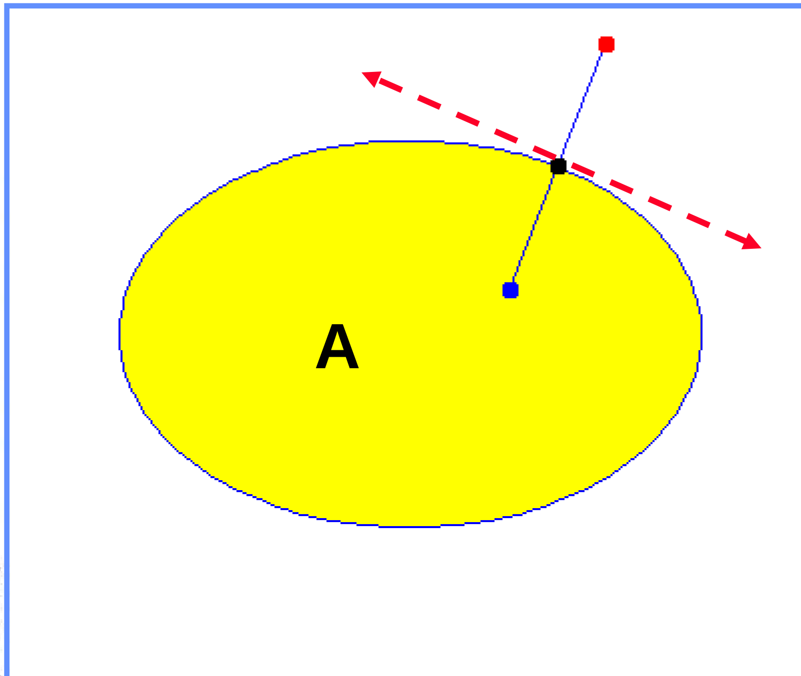
> "Why should I refuse a good dinner simply because I don't understand the digestive processes involved?" - Oliver Heaviside (1850-1925)

2. Phase Reconstruction

Projectors and Reflectors: $P_A(x)$ is the metric projection or nearest point and $R_A(x)$ reflects in the tangent: x is red

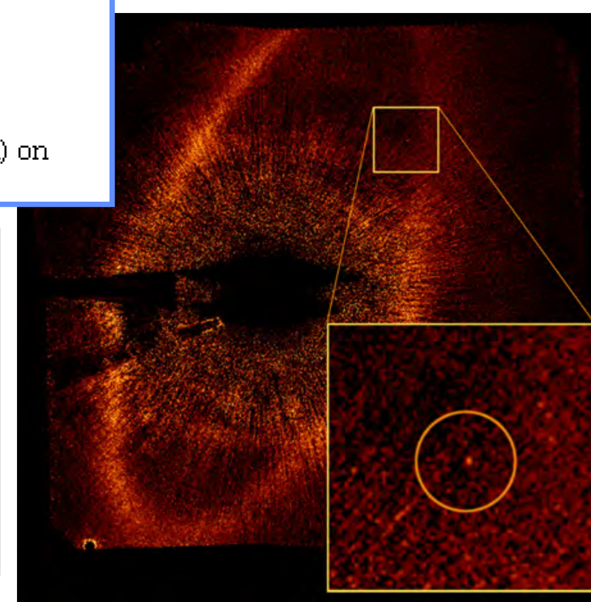
Veit Elser, Ph.D.

2007 Elser solving Sudoku with **reflectors**



projection (black) and reflection (blue) of point (red) on boundary (blue) of ellipse (yellow)

2008 Finding exoplanet Fomalhaut in Piscis with **projectors**



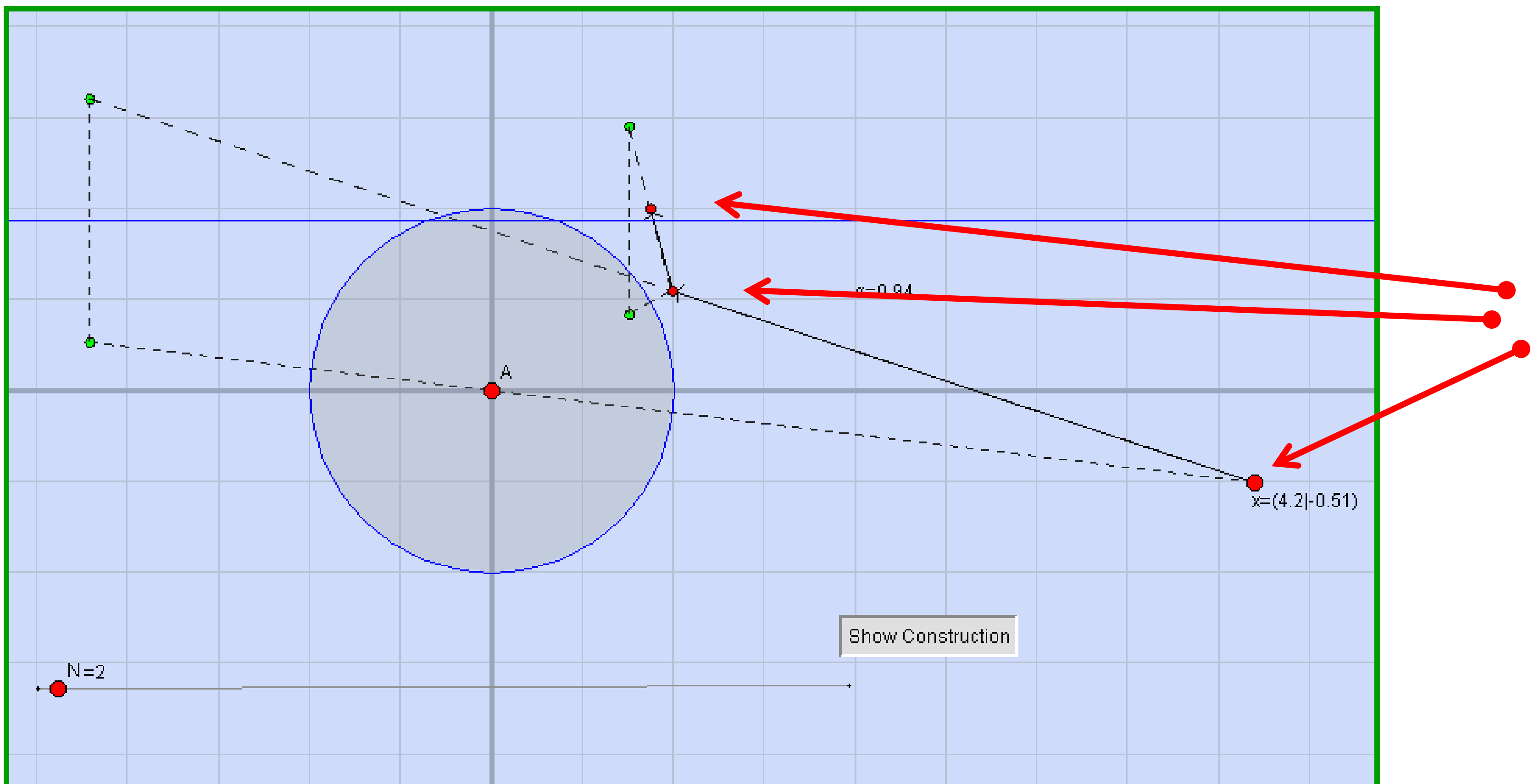
"All physicists and a good many quite respectable mathematicians are contemptuous about proof."
G. H. Hardy (1877-1947)

Interactive exploration in CINDERELLA

The simplest case is of a line A of height h and the unit circle B. With $z_n := (x_n, y_n)$ the iteration becomes

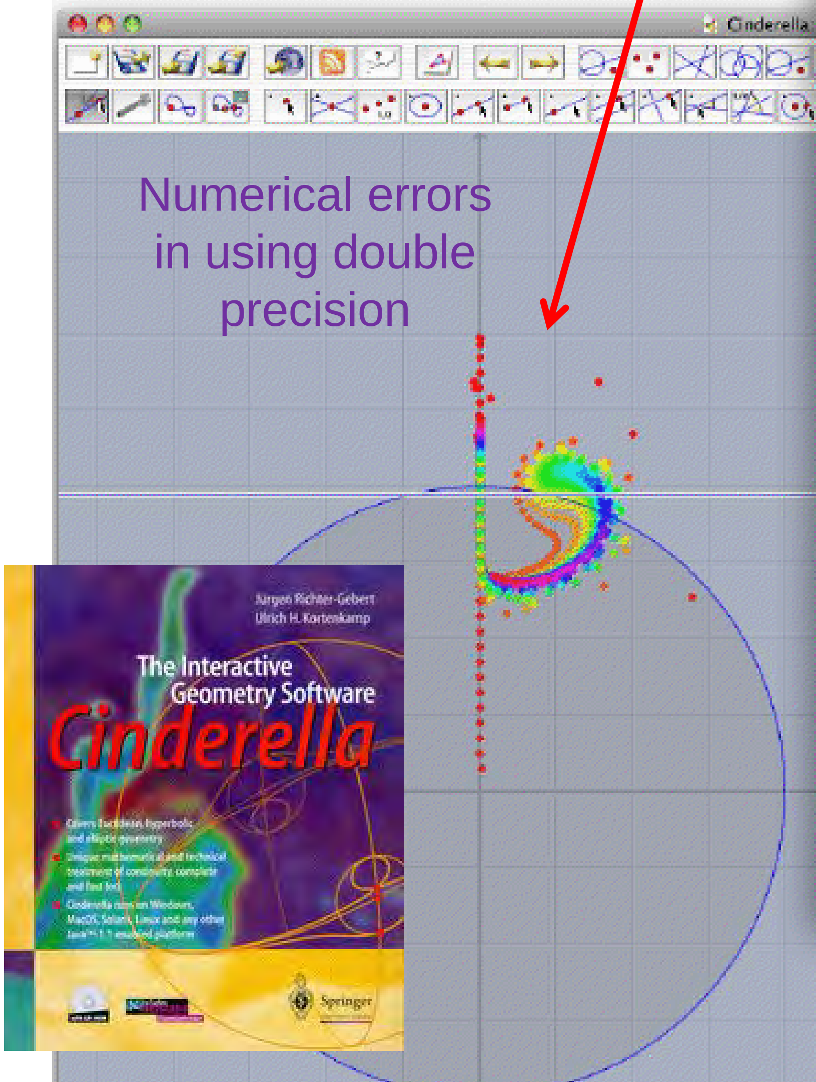
$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + h - \sin\theta_n, \qquad (\theta_n := \arg z_n)$$

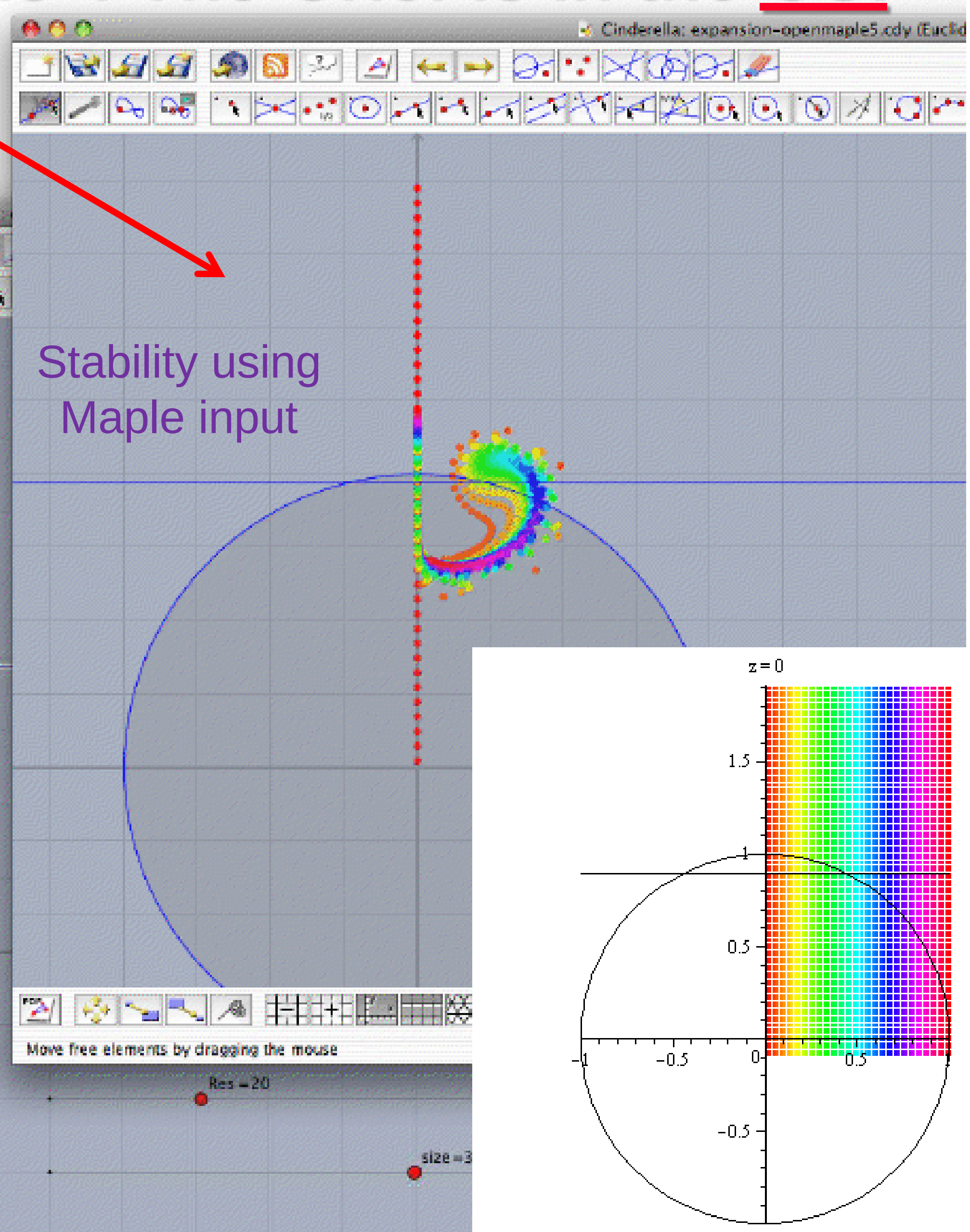
A *Cinderella* picture of two steps from (4.2,-0.51) follows:

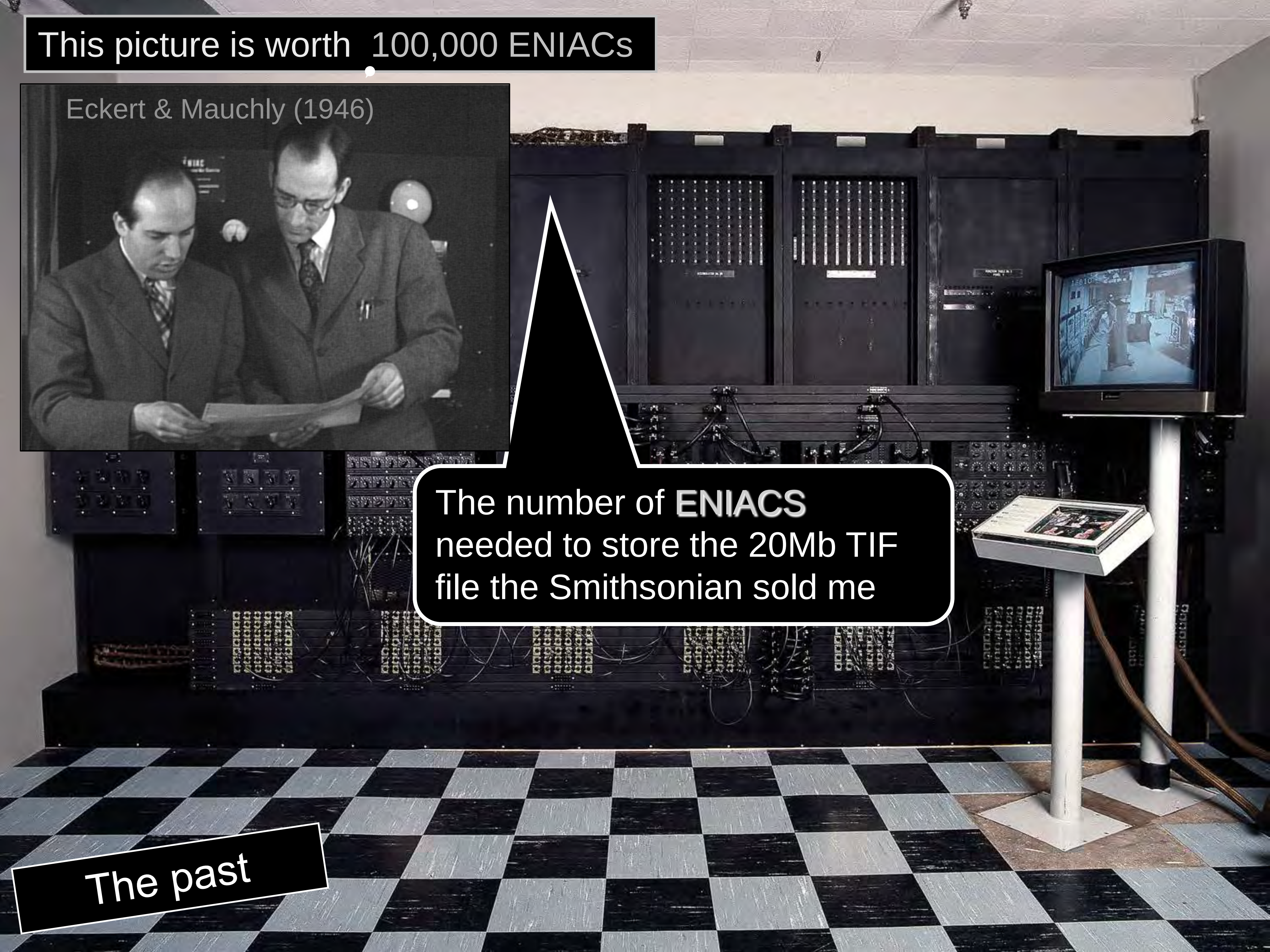


Computer Algebra + Interactive Geometry = "Visual Theorems": The Grief is in the GUI

Divide –and-Concur before and after accessing numerical output from **Maple**






This picture is worth 100,000 ENIACs
Eckert & Mauchly (1946)
The number of ENIACS needed to store the 20Mb TIF file the Smithsonian sold me
The past

PART II MATHEMATICS

“The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.” - Hermann Weyl

In “*Obituary: David Hilbert 1862 – 1943,*” *RSBIOS*, **4**, 1944, pp. 547-553; and *American Philosophical Society Year Book*, 1944, pp. 387-395, p. 392.

IIa. The Partition Function (1991-2009)

Consider the number of *additive* partitions, p(n), of n. Now

5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1

so p(5)=7. The ordinary generating function discovered by Euler is

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(1 - q^k\right)^{-1}. \qquad (1)$$

(Use the geometric formula for $1/(1-q^k)$ and observe how powers of q^n occur.)

The famous computation by MacMahon of **p(200)=3972999029388** done symbolically and entirely naively using (1) on an Apple laptop took **20 min** in **1991**, and about **0.17 seconds** in **2009.** Now it took **2 min** for **p(2000) = 4720819175619413888601432406799959512200344166**

In **2008**, Crandall found **p(10^9)** in **3 seconds** on a laptop, using the Hardy-Ramanujan-Rademacher 'finite' series for p(n) with FFT methods. Such fast partition-number evaluation let Crandall find *probable primes* **p(1000046356)** and **p(1000007396)**. Each has roughly 35,000 digits.

When does easy access to computation discourages innovation: would Hardy and Ramanujan have still discovered their marvellous formula for p(n)?

"YOU CAN'T IMAGINE HOW TIGHT OUR BUDGET IS. WE CAN ONLY WORK WITH SINGLE-DIGIT NUMBERS."

IIb. The computation of Pi (1986-2011)

$$y_1 = \frac{1-\sqrt[4]{1-y_0{}^4}}{1+\sqrt[4]{1-y_0{}^4}}, a_1 = a_0\left(1+y_1\right)^4 - 2^3 y_1\left(1+y_1+y_1{}^2\right)$$
$$y_2 = \frac{1-\sqrt[4]{1-y_1{}^4}}{1+\sqrt[4]{1-y_1{}^4}}, a_2 = a_1\left(1+y_2\right)^4 - 2^5 y_2\left(1+y_2+y_2{}^2\right)$$
$$y_3 = \frac{1-\sqrt[4]{1-y_2{}^4}}{1+\sqrt[4]{1-y_2{}^4}}, a_3 = a_2\left(1+y_3\right)^4 - 2^7 y_3\left(1+y_3+y_3{}^2\right)$$
$$y_7 = \frac{1-\sqrt[4]{1-y_6{}^4}}{1+\sqrt[4]{1-y_6{}^4}}, a_7 = a_6\left(1+y_7\right)^4 - 2^{15} y_7\left(1+y_7+y_7{}^2\right)$$
$$y_8 = \frac{1-\sqrt[4]{1-y_7{}^4}}{1+\sqrt[4]{1-y_7{}^4}}, a_8 = a_7\left(1+y_8\right)^4 - 2^{17} y_8\left(1+y_8+y_8{}^2\right)$$
$$y_9 = \frac{1-\sqrt[4]{1-y_8{}^4}}{1+\sqrt[4]{1-y_8{}^4}}, a_9 = a_8\left(1+y_9\right)^4 - 2^{19} y_9\left(1+y_9+y_9{}^2\right)$$
$$y_{10} = \frac{1-\sqrt[4]{1-y_9{}^4}}{1+\sqrt[4]{1-y_9{}^4}}, a_{10} = a_9\left(1+y_{10}\right)^4 - 2^{21} y_{10}\left(1+y_{10}+y_{10}{}^2\right)$$
$$y_{11} = \frac{1-\sqrt[4]{1-y_{10}{}^4}}{1+\sqrt[4]{1-y_{10}{}^4}}, a_{11} = a_{10}\left(1+y_{11}\right)^4 - 2^{23} y_{11}\left(1+y_{11}+y_{11}{}^2\right)$$
$$y_{12} = \frac{1-\sqrt[4]{1-y_{11}{}^4}}{1+\sqrt[4]{1-y_{11}{}^4}}, a_{12} = a_{11}\left(1+y_{12}\right)^4 - 2^{25} y_{12}\left(1+y_{12}+y_{12}{}^2\right)$$
$$y_{16} = \frac{1-\sqrt[4]{1-y_{15}{}^4}}{1+\sqrt[4]{1-y_{15}{}^4}}, a_{16} = a_{15}\left(1+y_{16}\right)^4 - 2^{33} y_{16}\left(1+y_{16}+y_{16}{}^2\right)$$
$$y_{17} = \frac{1-\sqrt[4]{1-y_{16}{}^4}}{1+\sqrt[4]{1-y_{16}{}^4}}, a_{17} = a_{16}\left(1+y_{17}\right)^4 - 2^{35} y_{17}\left(1+y_{17}+y_{17}{}^2\right)$$
$$y_{18} = \frac{1-\sqrt[4]{1-y_{17}{}^4}}{1+\sqrt[4]{1-y_{17}{}^4}}, a_{18} = a_{17}\left(1+y_{18}\right)^4 - 2^{37} y_{18}\left(1+y_{18}+y_{18}{}^2\right)$$
$$y_{19} = \frac{1-\sqrt[4]{1-y_{18}{}^4}}{1+\sqrt[4]{1-y_{18}{}^4}}, a_{19} = a_{18}\left(1+y_{19}\right)^4 - 2^{39} y_{19}\left(1+y_{19}+y_{19}{}^2\right)$$
$$y_{20} = \frac{1-\sqrt[4]{1-y_{19}{}^4}}{1+\sqrt[4]{1-y_{19}{}^4}}, \mathbf{a_{20}} = a_{19}\left(1+y_{20}\right)^4 - 2^{41} y_{20}\left(1+y_{20}+y_{20}{}^2\right).$$

These equations specify an algebraic number: $1/\pi \sim a_{20}$

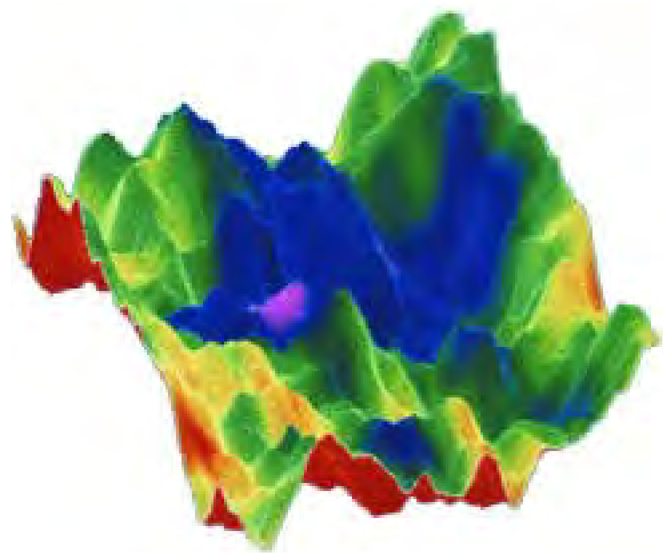
BB4: Pi to 5 trillion places in 21 steps

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \text{and}$$

$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

Then $1/a_k$ **converges quartically** to π



A random walk on a million digits of Pi

Moore' s Law Marches On

1986: It took Bailey 28 hours to compute **29.36 million digits** on 1 cpu of the then new CRAY-2 at NASA Ames using (BB4). Confirmation using another BB quadratic algorithm took 40 hours. This uncovered *hardware+software* errors on the CRAY.

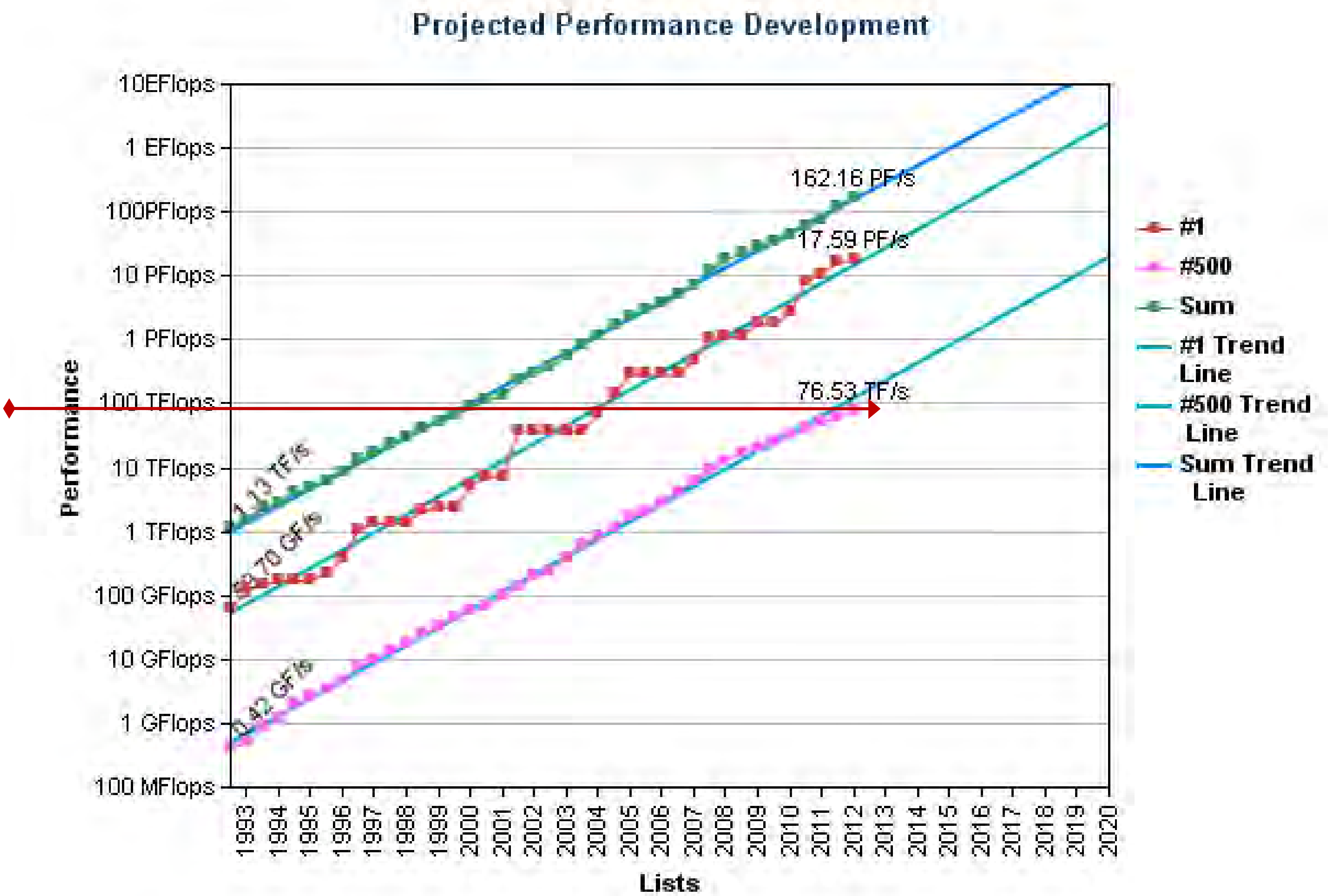
2009 Takahashi on 1024 cores of a 2592 core *Appro Xtreme - X3* system **1.649 trillion digits** via (Salamin-Brent) took 64 hours 14 minutes with 6732 GB of main memory, and (BB4) took 73 hours 28 minutes with 6348 GB of main memory.

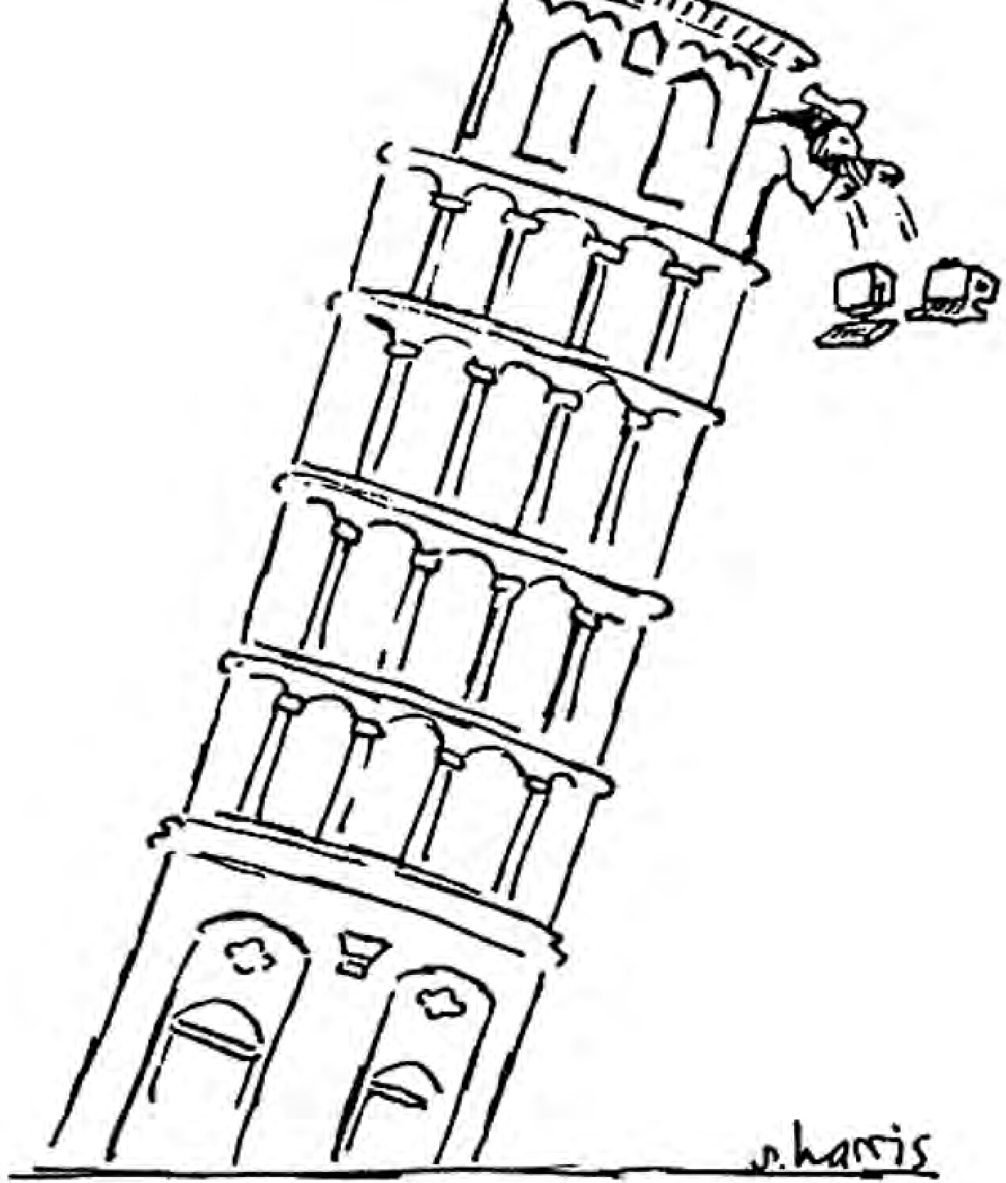
- The 2 computations differed only in last 139 places.

Fabrice Bellard (Dec 2009) **2.7 trillion places** on a 4 core desktop in 133 days after **2.59 trillion** by Takahashi (April).
2010/11: **Yee-Ohno** 5/**10 trillion digits** (my Lecture **Life of Pi**)

"***The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so***." - Leonard Carleson (*Lusin's problem* on p.w. convergence of Fourier series in Hilbert space)

Projected Performance





IF THERE WERE COMPUTERS IN GALILEO'S TIME

II c. Guiga and Lehmer (1932-2012)

As another measure of what changes over time and what doesn't, consider two conjectures regarding Euler's totient Á(n) which counts positive numbers less than and relatively prime to n.

Giuga's conjecture (1950) n is prime if and only if

$$\mathcal{G}_n := \sum_{k=1}^{n-1} k^{n-1} \equiv (n-1) \bmod n.$$

Counterexamples are *Carmichael numbers* (rare birds only proven infinite in **1994**) and more: if a number $n = p_1 \cdots p_m$ with m>1 prime factors p_i is a counterexample to Giuga's conjecture then the primes are distinct and satisfy

$$\sum_{i=1}^{m} \frac{1}{p_i} > 1$$

and they form a *normal sequence:* $p_i \neq 1 \bmod p_j$

(3 rules out 7, 13, 19,31,... and 5 rules out 11, 31, 41,...)

Guiga's Conjecture (1951-2012)

1995. With *predictive* experimentally-discovered heuristics, we built an efficient algorithm to show (in several months) that any counterexample had **3459** prime factors and so exceeded $\mathbf{10^{13886}}$

- $\rightarrow$ $\mathbf{10^{14164}}$ in a **5 day** desktop **2002** computation.
- Method fails after **8135** primes -- aim to exhaust it err I die.

2009. Almost as good a bound of **3050** primes in under **110** minutes on my Notebook and **3486** primes in **14 hours**:

- Not as before C++ which being compiled is faster but in which coding was much more arduous.
- Using one core of eight-core *MacPro* got **3592** primes and **16673** digits in **13.5 hrs** in *Maple.* (Now on 8 cores in 1 min of C++.)

2012/ 4771 prime factors, and excludes **19908** digits.

- Used C++, multithreaded on 8 cores of I7 core iMac. Took about a week but with 46 gigabyte output file.
- Time and especially file size now show massive exponential growth.

Lehmer's Conjecture (1932-2012)

Much tougher and related is

Lehmer's conjecture (1932) n is prime if and only if

$$\boxed{\phi(n)|(n-1)}$$

He called this "*as hard as the existence of odd perfect numbers.*"

- Again, prime factors of counterexamples form a normal sequence, but now there is little extra structure.

In a **1997** SFU M.Sc. Erick Wong verified this for **14** primes, using normality and a mix of PARI, C++ and Maple to press the bounds of the '*curse of exponentiality*.'

The related equation $\phi(n)\,|\,(n+1)$ is has **8** solutions with at most **7** factors (**6** factors is due to Lehmer).

- Recall $F_n := 2^{2^n}+1$ the *Fermat primes*. The solutions are 2, 3, 3.5, 3.5.17, 3.5.17.257, 3.5.17.257.65537 and a rogue pair: 4919055 and 69929626721320**95**, but **8** factors seems out of sight**.**
- Lehmer "couldn't" factor 6992962672132**97**= 73.95794009207289. If prime, a 9[th] would exist: $\phi(n)\,|\,(n+1)$ and n+2 prime => N:=n(n+2), $\phi(N)\,|\,(N+1)$

"Vacuums, black holes, antimatter - it's the elusive and intangible which appeals to me."

II d. Apéry-Like Summations

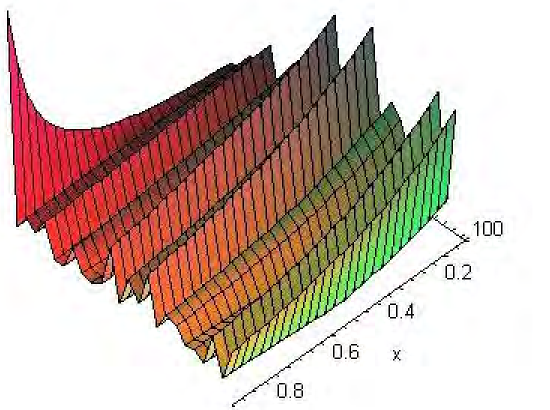
The following formulas for ζ(n) have been known for many decades:

$$(a) \quad \zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}},$$

$$(b) \quad \zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}},$$

$$(c) \quad \zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

The RH in Maple



These results have led many to speculate that

$$Q_5 := \zeta(5) / \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}$$

might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits.

> *"He (Gauss) is like the fox, who effaces his tracks in the sand with his tail."* - Niels Abel (1802-1829)

Two more things about Zeta(5)

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} = 2\zeta(5) - \tfrac{4}{3}L^5 + \tfrac{8}{3}L^3\zeta(2) + 4L^2\zeta(3)$$
$$+80 \sum_{n>0} \left(\frac{1}{(2n)^5} - \frac{L}{(2n)^4} \right) \rho^{2n}$$

Here $\rho := \frac{\sqrt{5}-1}{2}$ and $L := \log \rho$ (JMB-Broadhurst-Kamnitzer, 2000).

Also, there is a simpler Ramanujan series for $\zeta(4n+1)$. In particular:

$$\zeta(5) = \frac{1}{294}\pi^5 + \frac{2}{35} \sum_{k=1}^{\infty} \frac{1}{(1+e^{2k\pi})k^5} + \frac{72}{35} \sum_{k=1}^{\infty} \frac{1}{(1-e^{2k\pi})k^5},$$

and $\zeta(5) - \pi^5/294 = -0.0039555\ldots$.

Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \times \frac{\left(5(n+1)^2 + 6(a-1)(n+1) + 2(a-1)^2\right)}{\prod_{k=0}^{n} (a+k)^4}.$$

Note: *Maple* establishes this identity as

$$-1/2\,\Psi(2,a) = -1/2\,\Psi(2,a) - \zeta(3) + 5/4\,{}_4F_3([1,1,1,1],[3/2,2,2],-1/4)$$

Hence

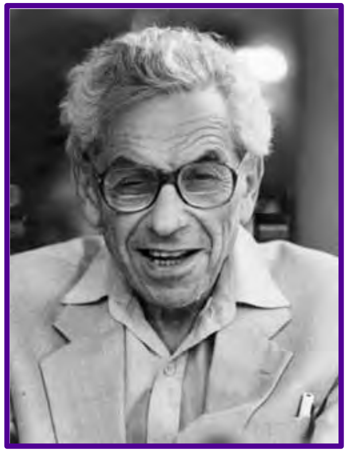
$$\zeta(4) = -\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2m}{m} m^4} + \frac{10}{3} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \sum_{k=1}^{m} \frac{1}{k}}{\binom{2m}{m} m^3}.$$

- The case a=0 above is Apéry's formula for $\zeta(3)$!

Andrei Andreyevich Markov
(1856-1922)

Two Discoveries: 1995 and 2005

- **Two computer-discovered generating functions**
  - (1) was 'intuited' by Paul Erdös (1913-1996)
- **and (2) was a designed experiment**
  - was proved by the computer (Wilf-Zeilberger)
  - and then by people (Wilf included)
  - What about 4k+1?



$$\sum_{k=0}^{\infty} \zeta(4k+3)\, x^{4k} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k} (1 - x^4/k^4)} \prod_{m=1}^{k-1} \left(\frac{1 + 4x^4/m^4}{1 - x^4/m^4} \right) \qquad (1)$$

x=0 gives (b) and (a) respectively

$$\sum_{k=0}^{\infty} \zeta(2k+2)\, x^{2k} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k} (1 - x^2/k^2)} \prod_{m=1}^{k-1} \left(\frac{1 - 4\, x^2/m^2}{1 - x^2/m^2} \right) \qquad (2)$$

Apéry summary

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

♦Euler (1707-73)

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$

1. via PSLQ to 5,000 digits (120 terms)

$$\begin{aligned} \mathcal{Z}(x) &= 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2} \\ &= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} \\ &= \frac{1 - \pi x \cot(\pi x)}{2x^2} \end{aligned}$$

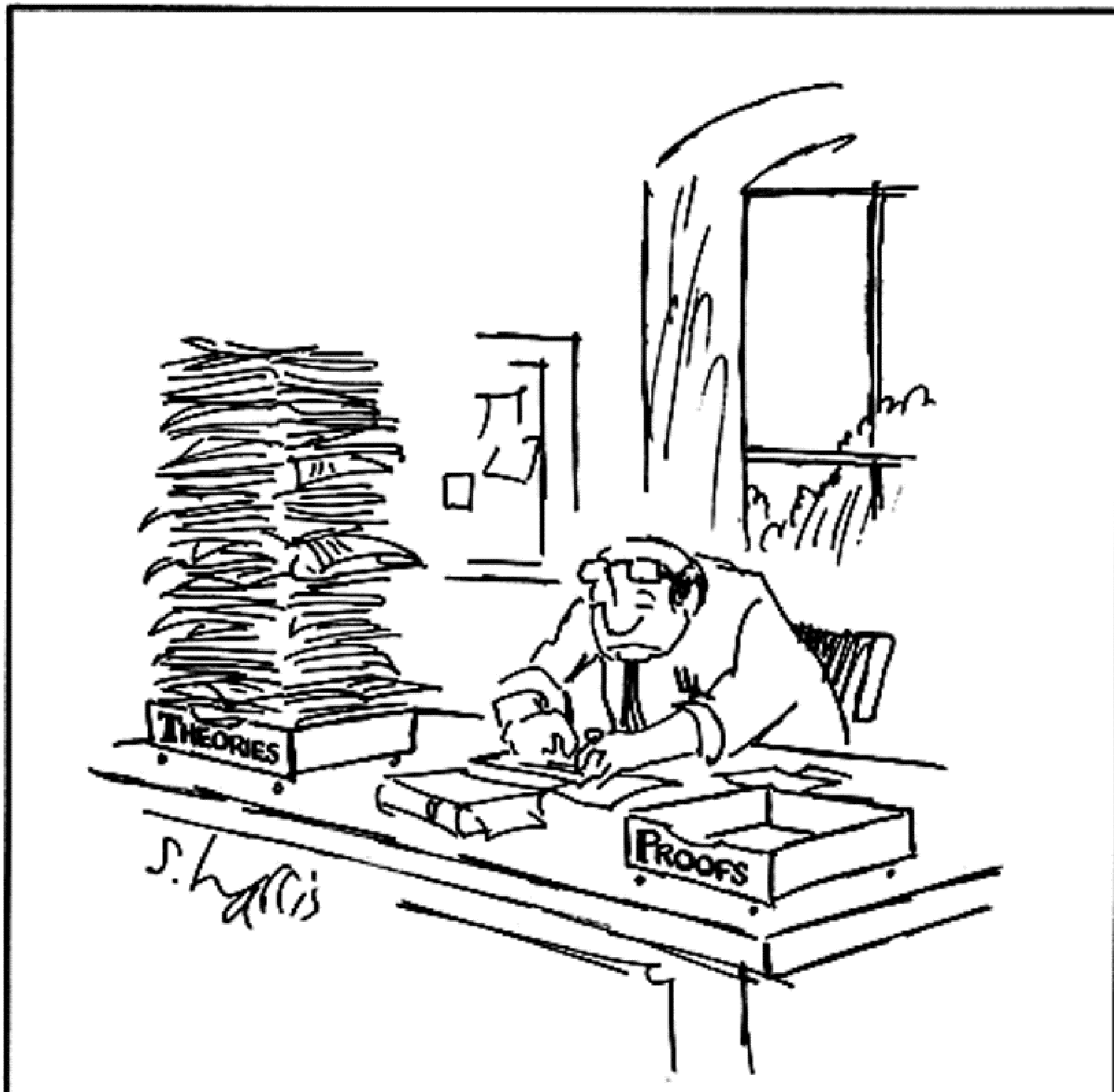
2. reduced as hoped

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}\left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

2005 Bailey, Bradley & JMB *discovered and proved* - in 3Ms - three *equivalent* binomial identities

$$_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven** (Wilf-Zeilberger) **(now 2 human proofs)**


THEORIES
PROOFS
S. Harris

II e: Ramanujan-Like Identities

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910. One is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}. \qquad (1)$$

Each term of (1) adds 8 correct digits. Gosper used (1) by the computation of a then-record 17 million digits of the c.f. for π in 1985— completing the first proof of (1).

A little later David and Gregory Chudnovsky found the following variant, which lies in $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \qquad (2)$$

Each term of (2) adds 14 correct digits.

- Used for current 10 trillion π-record.

They used (2) several times --- culminating in a 1994 calculation to over four billion decimal digits. Their remarkable story was told in a Pulitzer-winning New Yorker article.

New Ramanujan-Like Identities

Guillera has recently found Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}.$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. **It is true but has no proof.**

As far as we can tell there are no higher-order analogues!

Example of Use of Wilf-Zeilberger, I

The first two recent experimentally-discovered identities are

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}} \left(120n^2 + 34n + 3\right) = \frac{32}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2 + 180n + 13\right) = \frac{128}{\pi^2}$$

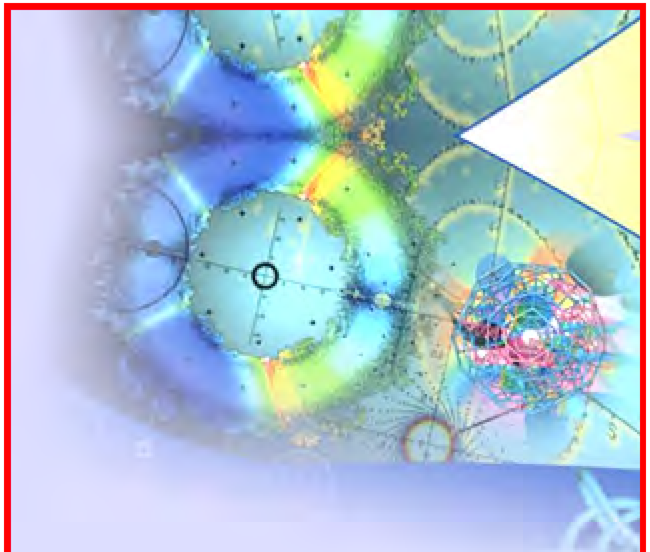
Guillera ***cunningly*** started by defining

$$G(n,k) = \frac{(-1)^k}{2^{16n}2^{4k}} \left(120n^2 + 84nk + 34n + 10k + 3\right) \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

He then used the **EKHAD** software package to obtain the companion

$$F(n,k) = \frac{(-1)^k 512}{2^{16n}2^{4k}} \frac{n^3}{4n-2k-1} \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

Wilf-Zeilberger, II



http://ddrive.cs.dal.ca/~isc/portal

When we define

$$H(n,k) \;=\; F(n+1,n+k)+G(n,n+k)$$

Zeilberger's theorem gives the identity

$$\sum_{n=0}^{\infty} G(n,0) \;=\; \sum_{n=0}^{\infty} H(n,0)$$

which when written out is

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4\binom{4n}{2n}}{2^{16n}}\left(120n^2+34n+3\right) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n+7}}\frac{(n+1)^3}{2n+3}\frac{\binom{2n+2}{n+1}^4\binom{2n}{n}^3\binom{2n+4}{n+2}}{\binom{2n+2}{n}\binom{2n+1}{n+1}^2}$$

$$+\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}}\left(204n^2+44n+3\right)\binom{2n}{n}^5 \;=\; \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}^5}{2^{20n}}\left(820n^2+180n+13\right)$$

A limit argument and **Carlson's theorem** completes the proof…

Searches for Additional Formulas

We had no PSLQ over number fields so we searched for additional formulas of either the following forms:

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}$$

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}.$$

where c is some linear combination of

$1,\ 2^{1/2},\ 2^{1/3},\ 2^{1/4},\ 2^{1/6},\ 4^{1/3},\ 8^{1/4},\ 32^{1/6},\ 3^{1/2},\ 3^{1/3},\ 3^{1/4},\ 3^{1/6},\ 9^{1/3},$
$27^{1/4},\ 243^{1/6},\ 5^{1/2},\ 5^{1/4},\ 125^{1/4},\ 7^{1/2},\ 13^{1/2},\ 6^{1/2},\ 6^{1/3},\ 6^{1/4},\ 6^{1/6},$
$7, 36^{1/3},\ 216^{1/4},\ 7776^{1/6},\ 12^{1/4},\ 108^{1/4},\ 10^{1/2},\ 10^{1/4},\ 15^{1/2}$

where each of the coefficients p_i is a linear combination of

$1,\ 2^{1/2},\ 3^{1/2},\ 5^{1/2},\ 6^{1/2},\ 7^{1/2},\ 10^{1/2},\ 13^{1/2},\ 14^{1/2},\ 15^{1/2},\ 30^{1/2}$

and where α is chosen as one of the following:

$1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128,\ 1/256,\ \sqrt{5}-2,\ (2-\sqrt{3})^2,$
$5\sqrt{13}-18,\ (\sqrt{5}-1)^4/128,\ (\sqrt{5}-2)^4,\ (2^{1/3}-1)^4/2,\ 1/(2\sqrt{2}),$
$(\sqrt{2}-1)^2,\ (\sqrt{5}-2)^2,\ (\sqrt{3}-\sqrt{2})^4$

Relations Found by PSLQ

- Including Guillera's three we found all known series for r(n) and no more.
- There are others for other Pochhammer symbols (JMB, Dec 2012 Notices)

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} r(n)^3(1+6n)\left(\frac{1}{2}\right)^{2n}$$

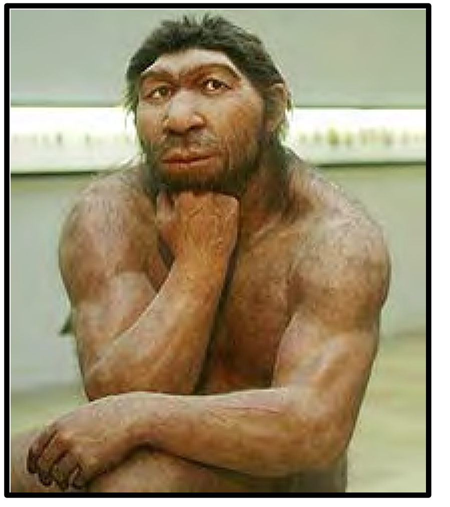
$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r(n)^3(5+42n)\left(\frac{1}{8}\right)^{2n}$$

$$\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-15+9\sqrt{3}-36n+24\sqrt{3}n)\left(2-\sqrt{3}\right)^{4n}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-1+5\sqrt{5}+30n+42\sqrt{5}n)\left(\frac{(\sqrt{5}-1)^4}{128}\right)^{2n}$$

$$\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-525+235\sqrt{5}-1200n+540\sqrt{5}n)\left(\sqrt{5}-2\right)^{8n}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(1+6n)\left(\frac{1}{2\sqrt{2}}\right)^{2n}$$



$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(-5+4\sqrt{2}-12n+12\sqrt{2}n)\left(\sqrt{2}-1\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(23-10\sqrt{5}+60n-24\sqrt{5}n)\left(\sqrt{5}-2\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(177-72\sqrt{6}+420n-168\sqrt{6}n)\left(\sqrt{3}-\sqrt{2}\right)^{8n}$$

Baruah, Berndt, Chan, "Ramanujan Series for 1/¼: A Survey." Aug 09, MAA *Monthly*

"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."

III. A Cautionary Example

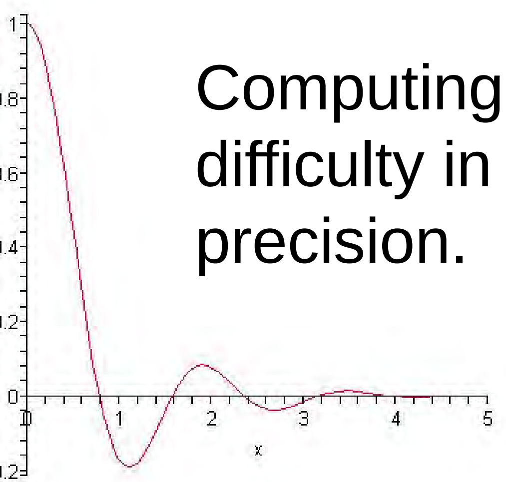
These **constants agree to 42 decimal digits** accuracy, but are **NOT** equal:

$$\int_0^\infty \cos(2x) \prod_{n=0}^{\infty} \cos(x/n)\, dx \quad =$$

$$0.39269908169872415480783042290993786052464\mathbf{543418723}\ldots$$

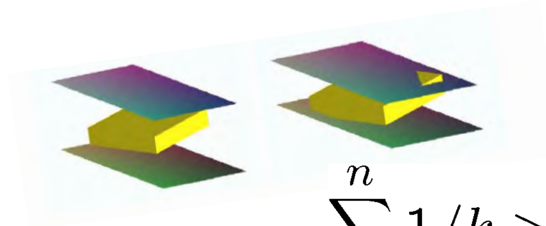
$$\frac{\pi}{8} \quad =$$

$$0.39269908169872415480783042290993786052464\mathbf{617492189}\ldots$$



Computing this integral is (or was) nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP) …



$$\sum_k^n 1/k > 2$$

IV. Some Conclusions

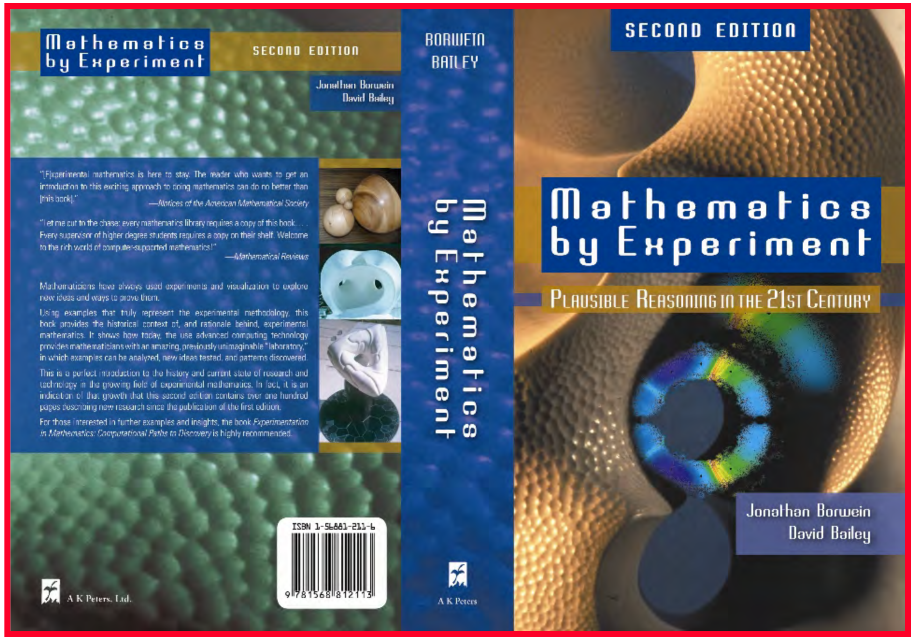
- We like students of **2012** live in an information-rich, judgement-poor world
- The explosion of information is not going to diminish
  - nor is the desire (need?) to collaborate remotely
- So we have to learn and teach judgement (not obsession with plagiarism)
  - that means mastering the sorts of tools I have illustrated
- We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (few future mathematicians)
  - properly balanced, discovery and proof can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion
- **Impediments** to the assimilation of the tools I have illustrated are myriad
  - as I am only too aware from recent experiences
- These impediments include our own inertia and
  - organizational and technical bottlenecks (IT - not so much dollars)
  - under-prepared or mis-prepared colleagues
  - the dearth of good modern syllabus material and research tools
  - the lack of a compelling business model (societal goods)

"***The plural of 'anecdote' is not 'evidence'.***"
- Alan L. Leshner (*Science's* publisher)

Further Conclusions

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof (say via WZ) when one "knows" the answer is right.



Full details of most examples are in *Mathematics by Experiment* or its companion volume *Experimentation in Mathematics* written with Roland Girgensohn. A "Reader's Digest" version of these is available at **www.experimentalmath.info** along with much other material.

"*Anyone who is not shocked by quantum theory has not understood a single word*." - Niels Bohr

An xkcd Farewell