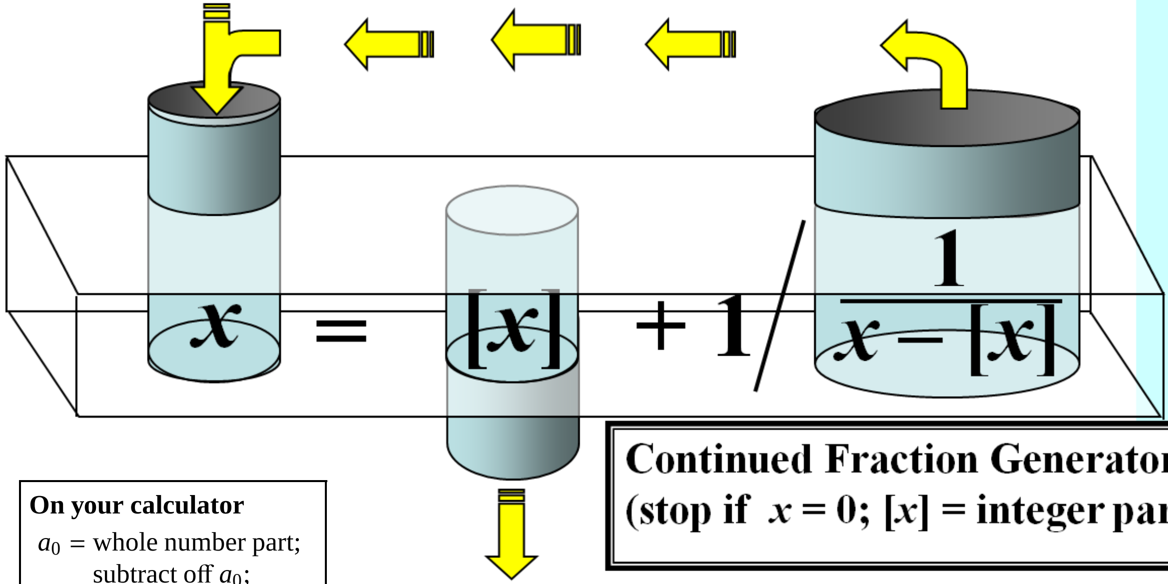
THEOREM OF THE DAY

Khinchin's Continued Fraction Theorem *There is a constant K such that, for almost all real numbers x, if x has continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ then the geometric means of the sets $\{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, converge to K as $n \to \infty$.*



Continued Fraction Generator
(stop if $x = 0$; $[x]$ = integer part of x)

$$\tau = 2\pi = 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{146 + 1/3.15163\ldots}}}}}}$$

$$= 6.283185307\ldots$$

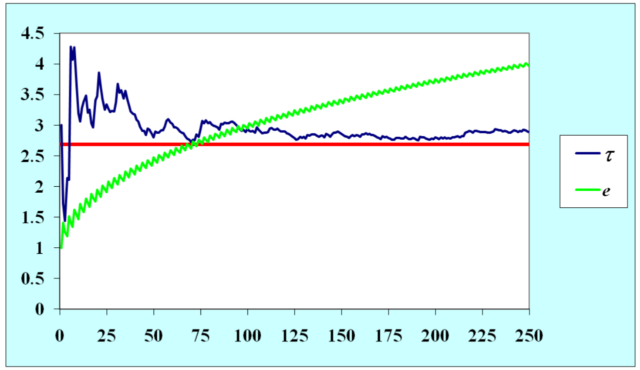
On your calculator

a_0 = whole number part;
subtract off a_0;
raise to power −1;
a_1 = whole number part;
subtract off a_1;
raise to power −1;
a_2 = etc, etc

E.g. $e = 2.71828\ldots$
$a_0 = 2$
$-2 \to 0.71828\ldots$
$\wedge -1 \to 1.39221\ldots$
$a_1 = 1$
$-1 \to 0.39221\ldots$
$\wedge -1 \to 2.54964\ldots$
$a_2 = 2$

Expressing a real number x as a continued fraction provides, among many other things, the best way of approximating x by rationals. For instance, truncating the expansion of $\tau = [6; 3, 1, 1, \ldots]$ after only three terms approximates τ to 1 decimal place as $6 + 1/(3 + 1) = 25/4$; by the sixth term (the fifth 'convergent') $6 + 1/(3 + 1/(1 + 1/(1 + 1/(7 + 1/2)))) = 710/113 \approx 6.28318584$, is already as accurate as rounding to the fifth decimal place (628318/100000).

Khinchin's 1935 theorem applies to 'almost all' real numbers in the technical sense, meaning 'to all real numbers except a set of measure zero'. The countable subset consisting of the rational numbers, for example, is excluded, as are surds, the golden ratio and the natural logarithm base e. As a matter of fact, *no* finitely specified real number has been shown to satisfy Khinchin's theorem, though an implicitly specified one is known (see the web link below). In the graph on the right, the geometric means $(\prod_{i=1}^{n} a_i)^{1/n}$ are plotted for τ and e, for n up to 250. It appears clear that e does *not* give convergence to $K \approx 2.6854520010$. The evidence for convergence for τ seems strong but, even after more than eighty years, still no proof is known.



Aleksandr Yakovlevich Khinchin (1894–1959) was an important Russian probability theorist who also made significant contributions to statistical mechanics and number theory.

Web link: Thomas Wieting's paper at www.ams.org/journals/proc/2008-136-03/

Further reading: *Continued Fractions* by Aleksandr Khinchin, Dover Publications, new edition 1997.

An Exercise: find over 100 digits of K here: oeis.org/A002210; construct the continued fraction; does K satisfy Khinchin's Theorem?