Lagrange’s Four-Squares Theorem

Any non-negative integer, \( n \), may be written as a sum of four squares:

\[ n = w^2 + x^2 + y^2 + z^2, \]

where \( w, x, y \) and \( z \) are non-negative integers (some of which may be zero.)

The Mathematical Porter insists on piling boxes on his trolley in a pyramid of square layers to a height of at most four. Lagrange’s theorem says this may be accomplished no matter how many boxes the porter has. Here, the pyramid illustrates \( 23 = 3^2 + 3^2 + 2^2 + 1^2 \).

But now the porter finds he has overlooked a box! How will he restack 24 boxes on his trolley?

The result was known to Diophantus of Alexandria and was first explicitly asserted by Bachet, who translated Diophantus’s \textit{Arithmetica} into Latin in 1621. Its proof required a hundred and fifty years of work by modern mathematicians, culminating in Lagrange’s complete proof of 1770. More generally, \textbf{Waring’s Problem} (solved affirmatively but non-constructively in 1909 by Hilbert) asks if any positive integer \( n \) can be written as a sum of at most a fixed number of \( k \)-th powers. For instance, any non-negative integer can be written as a sum of 9 cubes; thus \( 23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 \). Actually, this is one of only two numbers requiring 9 cubes (the other being 239) and it is still unknown whether, for large enough integers, 6 cubes might be enough (8042 is the largest integer known to require 7 cubes).

\textbf{Web link:}  \url{www.maths.lancs.ac.uk/~jameson/foursquares.pdf}