THEOREM OF THE DAY

The Eratosthenes–Legendre Sieve *Let $\pi(x)$ denote the number of primes not exceeding x, and $P(x)$ denote the product of all primes not exceeding x. Then*

$$\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{d \mid P(\sqrt{x})} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where $\mu(n)$ is the Möbius function defined for positive integers n by

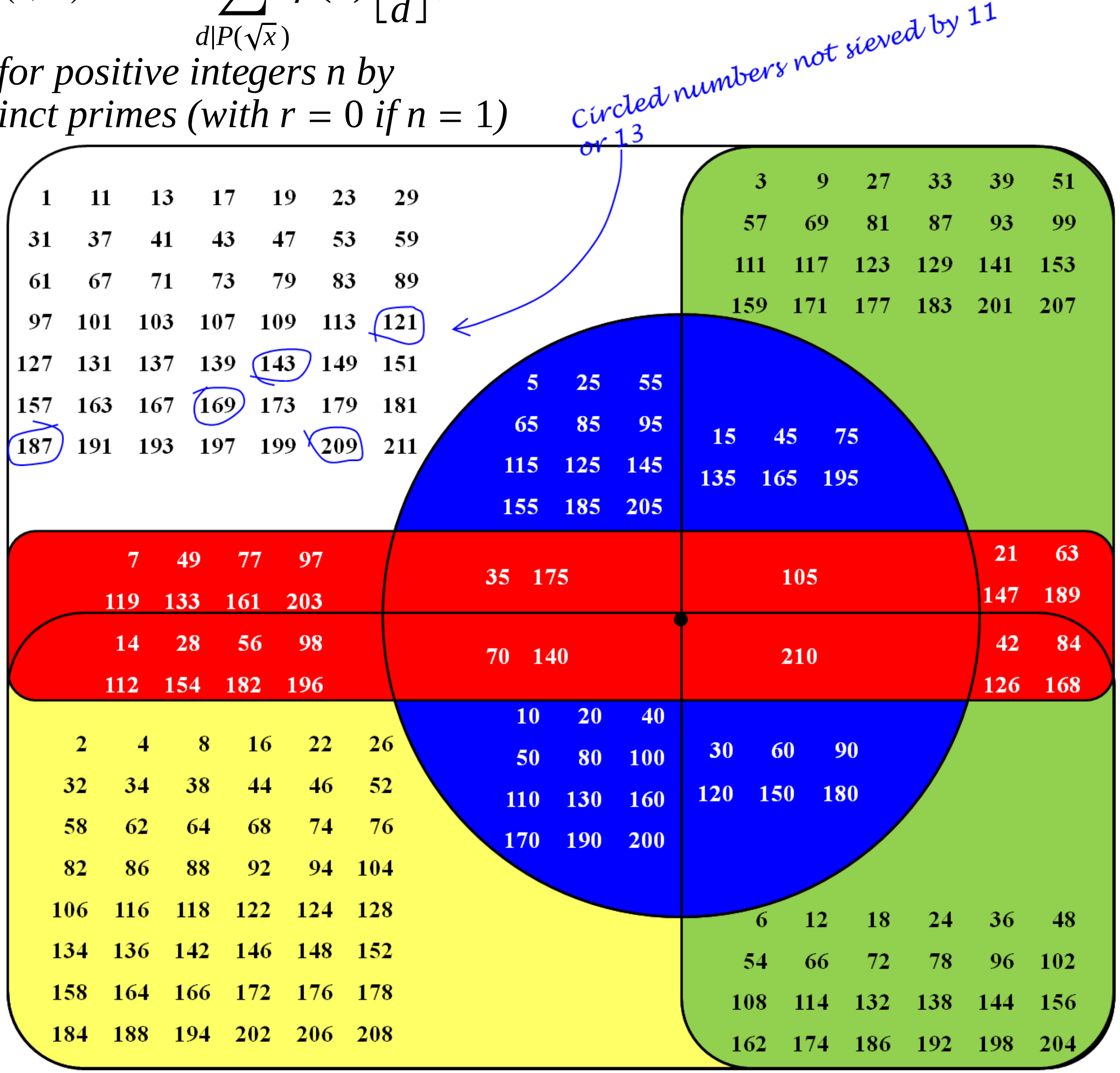
$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes (with } r = 0 \text{ if } n = 1) \\ 0 & \text{if } n \text{ has a square factor.} \end{cases}$$

How might we count the primes up to $211 = 1 + 2 \times 3 \times 5 \times 7$? A first approximation is to count all the integers from 1 to 211 which are excluded from the shaded regions of the 4-set Venn diagram on the right: bottom = multiples of 2; right = multiples of 3; circle=multiples of 5; central = multiples of 7. Inclusion-exclusion 'sieves out' products of just the first four primes (# denotes 'number of'):

			211
−	# multiples of 2, 3, 5, 7	$\left\lfloor \frac{211}{2} \right\rfloor + \ldots + \left\lfloor \frac{211}{7} \right\rfloor$	247
+	# multiples of $2 \times 3, 2 \times 5, \ldots, 5 \times 7$	$\left\lfloor \frac{211}{6} \right\rfloor + \ldots + \left\lfloor \frac{211}{35} \right\rfloor$	101
−	# multiples of $2 \times 3 \times 5, \ldots, 3 \times 5 \times 7$	$\left\lfloor \frac{211}{30} \right\rfloor + \ldots + \left\lfloor \frac{211}{105} \right\rfloor$	17
+	# multiples of $2 \times 3 \times 5 \times 7$	$\left\lfloor \frac{211}{210} \right\rfloor$	1
		Total (as shown in top-left of Venn diagram):	49

We have an estimate $\pi(211) - 4 + 1 \approx 49$, compensating for our four primes which have been sieved out, and counting the non-sieved non-prime 1. Legendre's version of Eratothenes' sieve is exact, extending inclusion-exclusion maximally to $\pi(\sqrt{211}) = |\{2, 3, 5, 7, 11, 13\}| = 6$, giving $\pi(211) - 6 + 1 = 42$. The Möbius function cleverly converts the alternating double sum of inclusion-exclusion into a single sum (Our summation is over all positive integers d which divide $P(\sqrt{x})$; but only $d \leq x$ count since $\lfloor x/d \rfloor$, the greatest integer not exceeding x/d, becomes zero when $d > x$.)

Eratosthenes, around 100BC, is credited with inventing the method of listing primes by sieving. Its adaptation by Legendre in 1808 to count primes is conceptually behind all modern sieve-based methods in number theory.



Web link: assets.press.princeton.edu/chapters/s8585.pdf: click on the book image for Ch. 1 of Glyn Harman's *Prime-Detecting Sieves*.
Further reading: *Sieve Methods* by Heini Halberstam and Hans-Egon Richert, Dover, 2011.