



# THEOREM OF THE DAY

**The Eratosthenes–Legendre Sieve** Let  $\pi(x)$  denote the number of primes not exceeding  $x$ , and  $P(x)$  denote the product of all primes not exceeding  $x$ . Then

$$\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{d|P(\sqrt{x})} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where  $\mu(n)$  is the Möbius function defined for positive integers  $n$  by

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes (with } r = 0 \text{ if } n = 1) \\ 0 & \text{if } n \text{ has a square factor.} \end{cases}$$

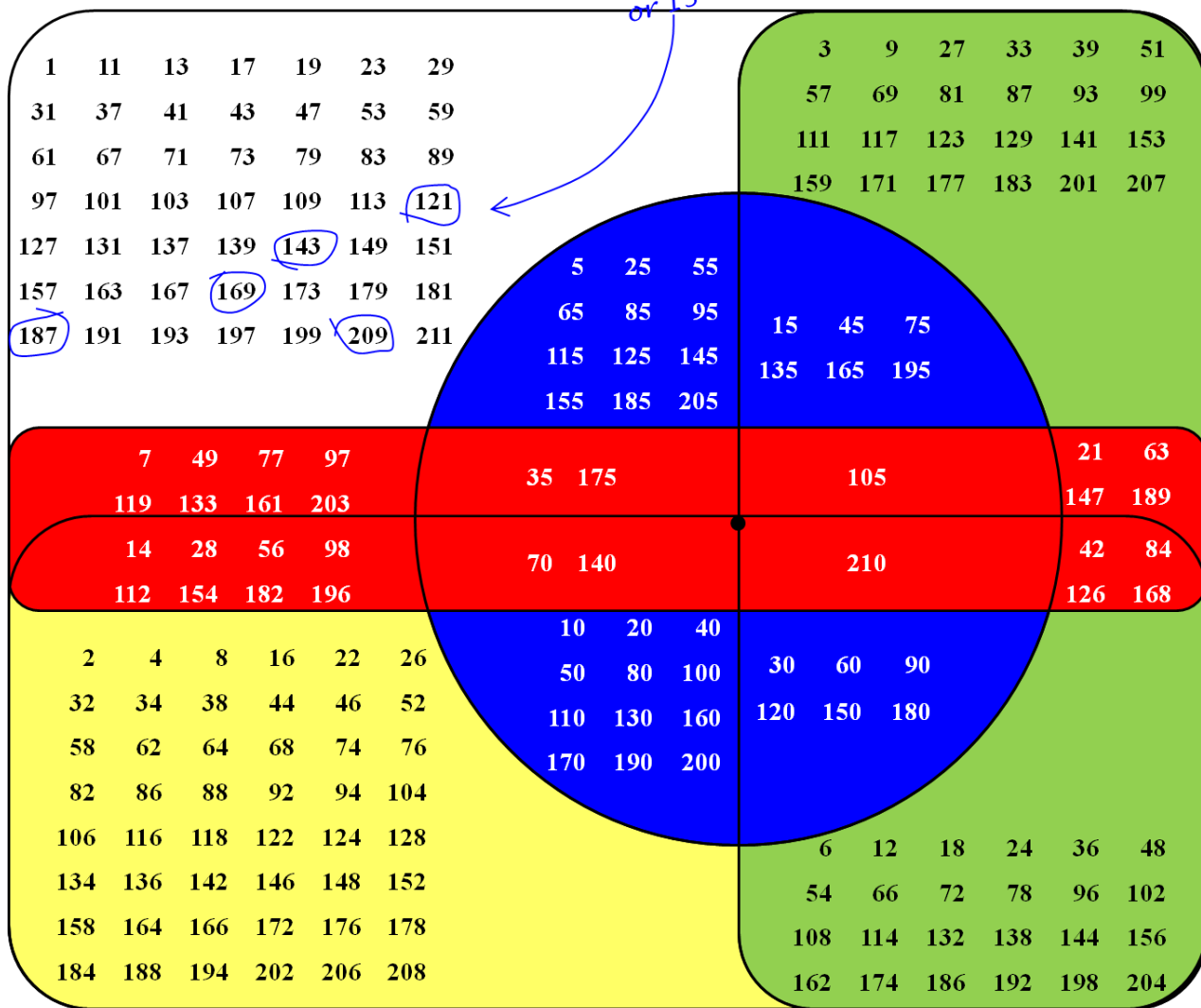
*Circled numbers not sieved by 11 or 13*

How might we count the primes up to  $211 = 1 + 2 \times 3 \times 5 \times 7$ ? A first approximation is to count all the integers from 1 to 211 which are excluded from the shaded regions of the 4-set Venn diagram on the right: bottom = multiples of 2; right = multiples of 3; circle = multiples of 5; central = multiples of 7. Inclusion-exclusion ‘sieves out’ products of just the first four primes (# denotes ‘number of’):

			211
-	# multiples of 2, 3, 5, 7	$\left\lfloor \frac{211}{2} \right\rfloor + \dots + \left\lfloor \frac{211}{7} \right\rfloor$	247
+	# multiples of $2 \times 3, 2 \times 5, \dots, 5 \times 7$	$\left\lfloor \frac{211}{6} \right\rfloor + \dots + \left\lfloor \frac{211}{35} \right\rfloor$	101
-	# multiples of $2 \times 3 \times 5, \dots, 3 \times 5 \times 7$	$\left\lfloor \frac{211}{30} \right\rfloor + \dots + \left\lfloor \frac{211}{105} \right\rfloor$	17
+	# multiples of $2 \times 3 \times 5 \times 7$	$\left\lfloor \frac{211}{210} \right\rfloor$	1
Total (as shown in top-left of Venn diagram):			49

We have an estimate  $\pi(211) - 4 + 1 \approx 49$ , compensating for our four primes which have been sieved out, and counting the non-sieved non-prime 1. Legendre’s version of Eratosthenes’ sieve is exact, extending inclusion-exclusion maximally to  $\pi(\sqrt{211}) = |\{2, 3, 5, 7, 11, 13\}| = 6$ , giving  $\pi(211) - 6 + 1 = 42$ . The Möbius function cleverly converts the alternating double sum of inclusion-exclusion into a single sum (Our summation is over all positive integers  $d$  which divide  $P(\sqrt{x})$ ; but only  $d \leq x$  count since  $\lfloor x/d \rfloor$ , the greatest integer not exceeding  $x/d$ , becomes zero when  $d > x$ .)

Eratosthenes, around 100BC, is credited with inventing the method of listing primes by sieving. Its adaptation by Legendre in 1808 to count primes is conceptually behind all modern sieve-based methods in number theory.



**Web link:** [assets.press.princeton.edu/chapters/s8585.pdf](https://assets.press.princeton.edu/chapters/s8585.pdf) (Chapter 1 of Glyn Harman’s *Prime-Detecting Sieves*, 1MB).

**Further reading:** *Sieve Methods* by Heini Halberstam and Hans-Egon Richert, Dover, 2011.