THEOREM OF THE DAY

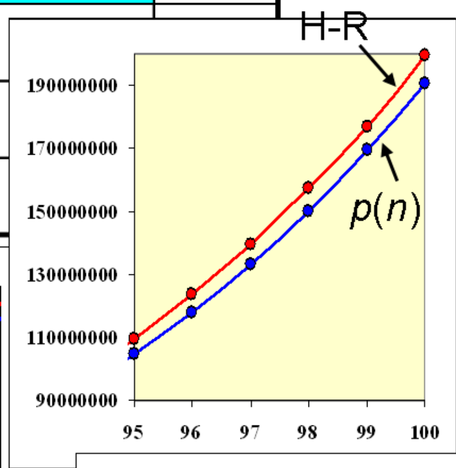
The Hardy–Ramanujan Asymptotic Partition Formula *For n a positive integer, let $p(n)$ denote the number of unordered partitions of n, that is, unordered sequences of positive integers which sum to n; then the value of $p(n)$ is given asymptotically by*

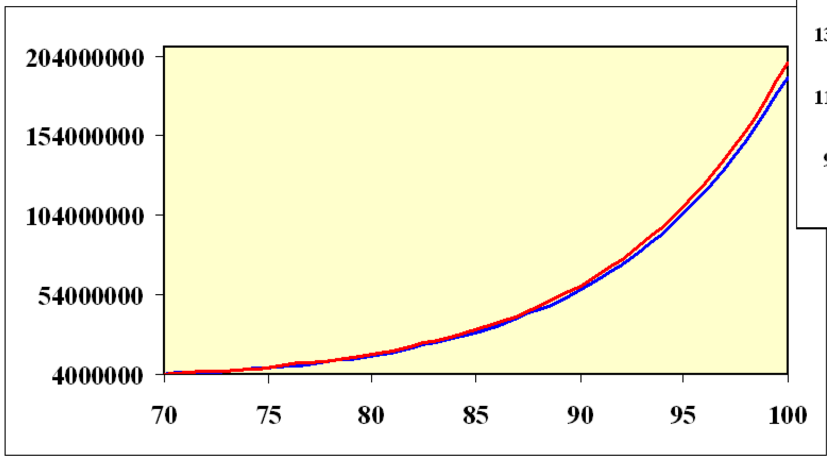
$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\tau\sqrt{n/6}}.$$

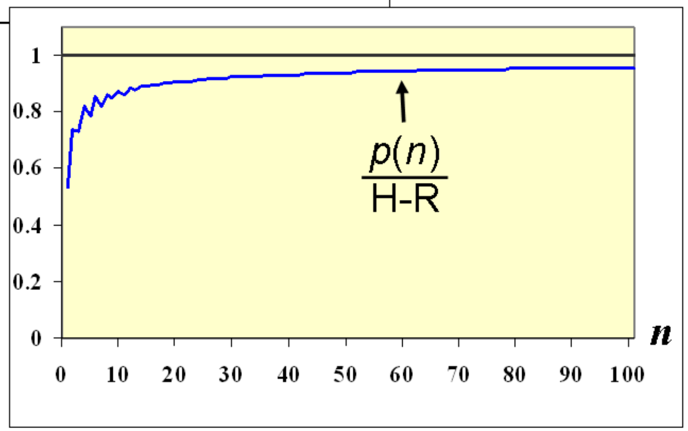
$\tau = 2\pi$ and e in another magical combination!

The value of $p(7)$ is 15, the partitions being displayed in the grid on the right. Note that they are 'unordered', in the sense that 1 + 6, say, is regarded as the same as 6 + 1 and is omitted. By $n = 70$, the number has risen to over 4 million; the Hardy-Ramanujan asymptotic estimate closely shadows this growth (the graphs below centre and right). Exactly how closely, is seen by taking the ratio $p(n)$/H-R (bottom right), the lower curve meeting the line $y = 1$ at infinity.

7	6+1	5+2	5+1+1	4+3
4+2+1	4+1+1+1	3+2+2	3+3+1	
3+2+1+1	3+1+1+1+1	2+2+2+1		
2+2+1+1+1	2+1+1+1+1+1			
1+1+1+1+1+1+1				







Srinivasa Ramanujan (1887–1920) had already made deep discoveries about the partition function when he was himself 'discovered' by G.H. Hardy in 1913. Five years later, despite Ramanujan being already terminally ill, they published this remarkable asymptotic formula, derived from a series which yields the *exact value* of $p(n)$ in $\sqrt{n}$ terms. It was discovered independently, in 1920, by James Victor Uspensky.

Web link: www.ias.ac.in/describe/article/reso/001/12/0081-0091: Atle Selberg talks about Ramanujan's life and work; his study of $p(n)$ is discussed on pp. 86–87.

Further reading: *Integer Partitions, 2nd revised ed.* by George E. Andrews and Kimmo Eriksson, Cambridge University Press, 2004.