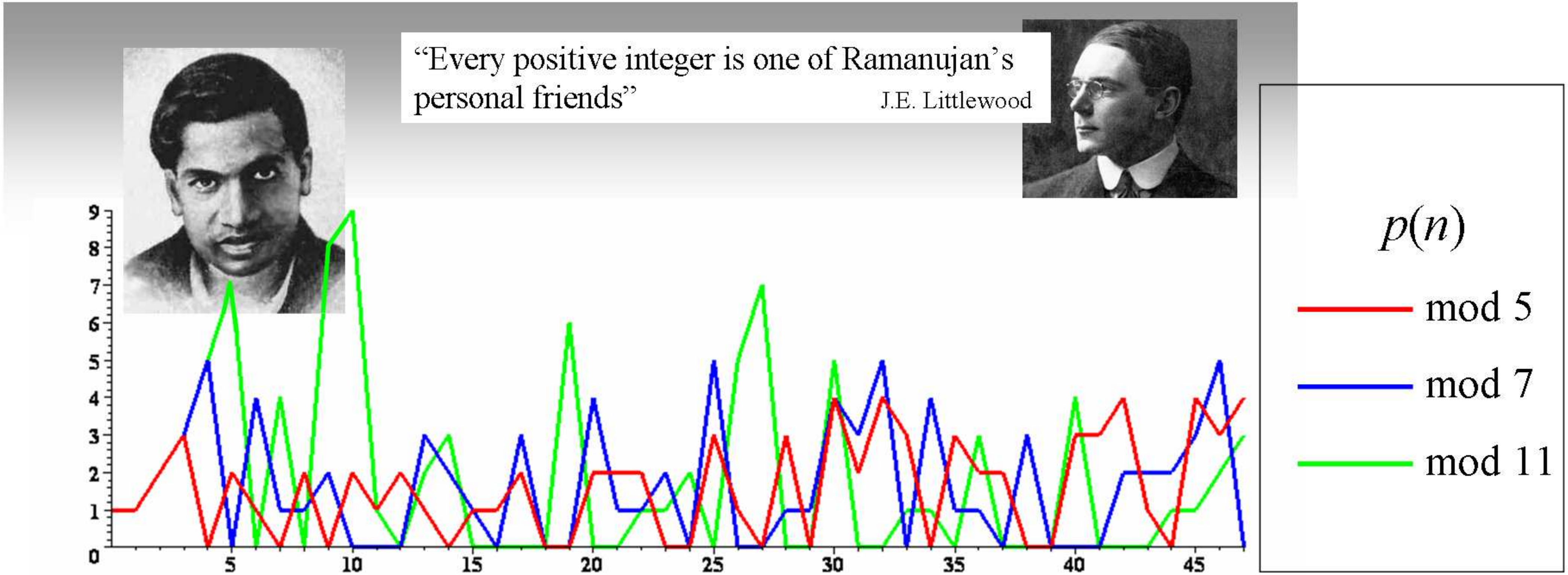
THEOREM OF THE DAY

The Ramanujan Partition Congruences *Let n be a non-negative integer and let p(n) denote the number of partitions of n (that is, the number of ways to write n as a sum of positive integers). Then p(n) satisfies the congruence relations:*

$$p(5t+4) \equiv 0 \pmod{5}, \qquad p(7t+5) \equiv 0 \pmod{7}, \quad \textit{and} \quad p(11t+6) \equiv 0 \pmod{11}.$$



Ramanujan’s congruences tell us that, in the set of values of n for which $p(n) \mod q = 0$, when q is 5, 7 or 11, there is an infinite arithmetic progression of common difference q. Thus we see that, in the above plot, the three graphs touch the horizontal axis at intervals which appear quite irregular but are certainly constrained by this arithmetic progression property. The property extends to all primes $q \geq 5$, a deep result published in 2000 by Ken Ono, but the common differences will not generally be q: the set of values of n for which $p(n) \mod 31 = 0$, for instance, contains an infinite arithmetic progression whose common difference is not 31 but 31×107^4, and which starts at $n = 30064597$. For $q = 3$, the situation is very different—it is not even known if the values of n for which $p(n) \mod 3 = 0$ form an infinite set!

Ramanujan published proofs of the congruences for 5 and 7 in 1919. His proof for mod 11 remained unpublished at the time of his death in 1920 and was written up by G.H. Hardy.

Web link: web.maths.unsw.edu.au/~mikeh/webpapers/paper171.pdf. The wonderful biographies at www-history.mcs.st-and.ac.uk (whence the above pictures of Ramanujan and J.E. Littlewood) link to selections of quotations: thus you may discover the origin of Littlewood’s remark.

Further reading: *Integer Partitions, 2nd revised ed.* by George E. Andrews and Kimmo Eriksson, Cambridge University Press, 2004