THEOREM OF THE DAY

The Riemann Explicit Formula *The number of primes not exceeding a given real number x is given by*

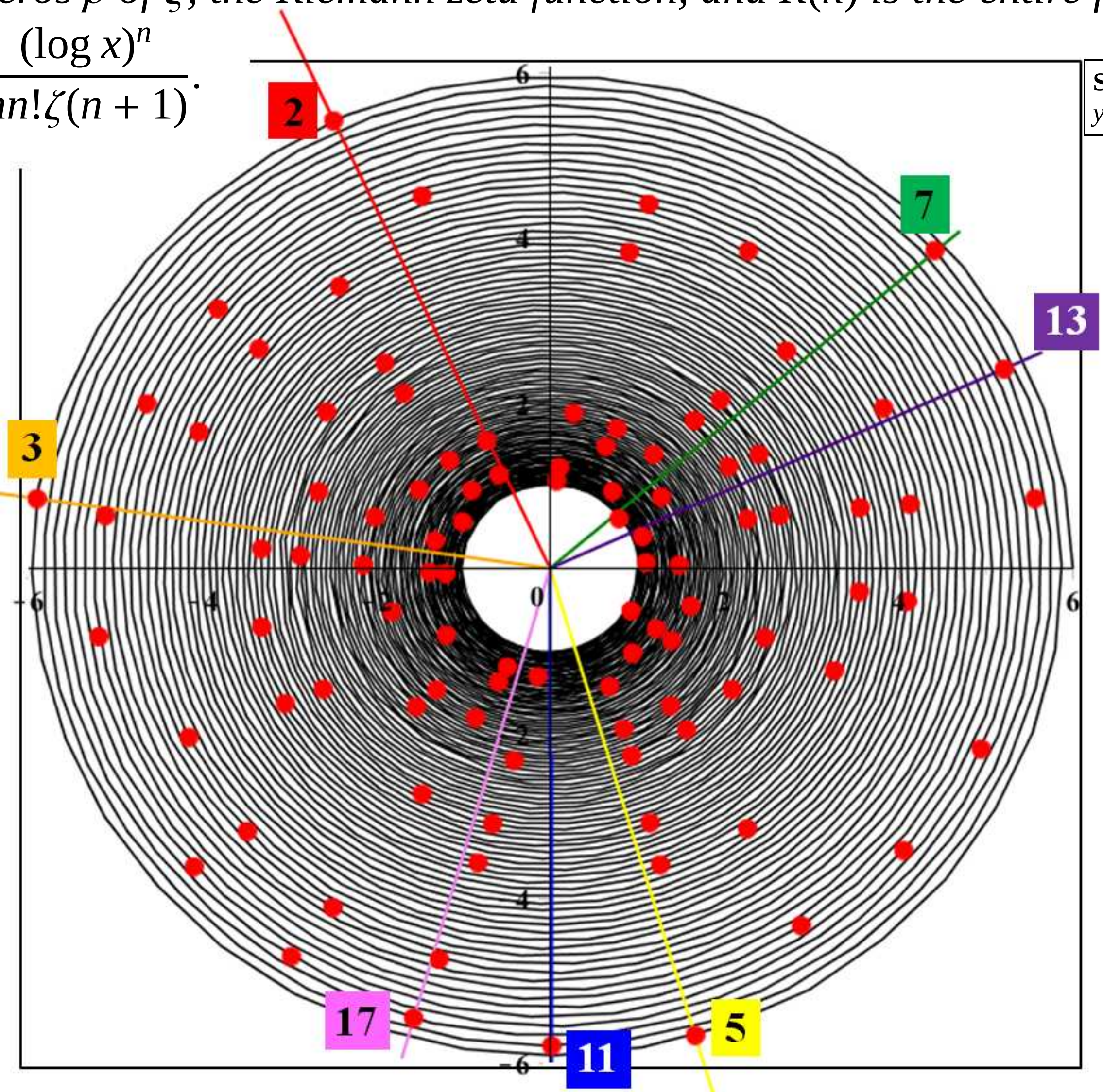
$$\pi(x) = R(x) + \sum R\left(x^{\rho}\right),$$

where the sum is over all zeros ρ of ζ, the Riemann zeta function, and $R(x)$ is the entire function of $\log x$ *defined by* $R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n n! \zeta(n+1)}$.

Riemann's formula originates in Gauss's empirical observation that the frequency of primes in the vicinity of a number N is 'close' to $1/\log N$. Summing for N between 2 and x and taking the continuous, calculus version gives the approximation $\int_2^x 1/\log t\,dt \approx \pi(x)$. And indeed the logarithmic integral, denoted by $\mathrm{Li}(x)$, approximates $\pi(x)$ with an error which vanishes proportional to $\pi(x)$, as x gets large (the Prime Number Theorem). Riemann found a better approximation:

$$\mathrm{Li}(x) \approx \pi(x) + \tfrac{1}{2}\pi\left(x^{1/2}\right) + \tfrac{1}{3}\pi\left(x^{1/3}\right) + \ldots, \quad \textbf{(1)}$$

(the summation being finite since $\pi\left(x^{1/k}\right) = 0$, for $x^{1/k} < 2$). This is very accurate: for example, taking just the first 100 primes $2, 3, 5, \ldots, 541$, (shown on the right distributed around a logarithmic spiral starting on the outside, with each integer represented by 1 radian of angle) the integral is $\mathrm{Li}(541) = \int_2^{541} 1/\log t\,dt = 107.304\ldots$ while Riemann's summation is $100 + \frac{9}{2} + \frac{4}{3} + \frac{2}{4} + \frac{2}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} = 107.278\ldots$. This accuracy is due to Riemann's observation, that k-th powers of all primes $\leq x^{1/k}$ also populate the locality of N, for every k-th N. Thus the summation in (1).



Spiral equation: $x = e^{\lambda/48\tau}\cos(\lambda)$, $y = -e^{\lambda/48\tau}\sin(\lambda)$, $\lambda = 0\ldots\lfloor 541/\tau\rfloor\tau$.

The approximation (1)

$$\mathrm{Li}(x) \approx \textstyle\sum_{k\geq 1}\frac{1}{k}\pi\left(x^{1/k}\right)$$

is 'inverted' using the technique of Möbius inversion:

$$\pi(x) = \textstyle\sum_{k\geq 1}\frac{\mu(k)}{k}\mathrm{Li}\left(x^{1/k}\right), \quad \textbf{(2)}$$

where $\mu(n)$ is the Möbius function: $\mu(n) = (-1)^r$ when n is a product of r distinct primes, with $n = 1$ giving $r = 0$, while $\mu(n) = 0$ if n has a square factor.

The right-hand-side of (2) is precisely the Riemann function $R(x)$, reformulated as a power series in $\log(x)$ in our statement of Riemann's formula. It converges rapidly and usually approximates $\pi(x)$ very accurately: for our example, $R(541)$ converges to 100, to the nearest integer, after only 14 terms.

Riemann's final stroke of genius removed even this error, specifying it exactly in terms of the zeros of the ζ function.

Riemann's formula appears in his single monumental paper on number theory dated 1859. His function $R(x)$ was discovered independently by Ramanujan. The power series version of $R(x)$ is due to Jorgen Gram (1884).

Web link: empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/encoding1.htm

Further reading: *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics,* by John Derbyshire, Plume, 2004.