**THEOREM OF THE DAY**

The Riemann Explicit Formula: The number of primes not exceeding a given real number \( x \) is given by

\[
\pi(x) = R(x) + \sum \frac{\log x^n}{nn!\zeta(n+1)}.
\]

where the sum is over all zeros \( \rho \) of \( \zeta \), the Riemann zeta function, and \( R(x) \) is the entire function of \( \log x \) defined by

\[
R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{nn!\zeta(n+1)}.
\]

Riemann's formula originates in Gauss's empirical observation that the frequency of primes in the vicinity of a number \( N \) is 'close' to \( 1/\log N \). Summing for \( N \) between 2 and \( x \) and taking the continuous, calculus version gives the approximation \( \int_{2}^{x} \frac{1}{\log t} dt \approx \pi(x) \). And indeed the logarithmic integral, denoted by \( \text{Li}(x) \), approximates \( \pi(x) \) with an error which vanishes proportional to \( \pi(x) \), as \( x \) gets large (the Prime Number Theorem). Riemann found a better approximation:

\[
\text{Li}(x) \approx \pi(x) + \frac{x}{2\pi} \left( \log x^2 + \frac{\log x}{3} \right) + \ldots,
\]

(1)

(the summation being finite since \( \pi(x^{1/k}) = 0 \), for \( x^{1/k} < 2 \)). This is very accurate: for example, taking just the first 100 primes 2, 3, 5, ..., 541, (shown on the right distributed around a logarithmic spiral starting on the outside, with each integer represented by 1 radian of angle) the integral is \( \text{Li}(541) = \int_{2}^{541} \frac{1}{\log t} dt \approx 107,304 \ldots \) while Riemann's summation is \( 100 + \frac{9}{2} + \frac{9}{3} + \frac{9}{5} + \frac{9}{7} + \frac{9}{11} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} = 107,278 \ldots \) . This accuracy is due to Riemann's observation, that \( k\text{-th} \) powers of all primes \( \leq x^{1/k} \) also populate the locality of \( N \), for every \( k\text{-th} \) \( N \). Thus the summation in (1).

Riemann's formula appears in his single monumental paper on number theory dated 1859. His function \( R(x) \) was discovered independently by Ramanujan. The power series version of \( R(x) \) is due to Jorgen Gram (1884).

Web link: emplocal.ex.ac.uk/people/staff/mrwatkin/zeta/encoding1.htm


Created by Robin Whitty for www.theoremoftheday.org