



# THEOREM OF THE DAY

**The Robin–Lagarias Theorem** Let  $H_n$  denote the  $n$ -th harmonic number  $\sum_{i=1}^n \frac{1}{i}$ , and let  $\sigma(n)$  denote the divisor function  $\sum_{d|n} d$ . Then the Riemann Hypothesis is equivalent to the statement that, for  $n \geq 1$ ,

$$\sigma(n) \leq H_n + \ln(H_n) e^{H_n}.$$

E.g.  $\sigma(6) = 1 + 2 + 3 + 6 = 12$

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45$$

$$2.45 + e^{2.45} \times \ln(2.45) \approx 2.45 + 11.59 \times 0.90 \approx 12.88.$$

The zeta function  $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ , for integer  $s \geq 2$ , has a close link with the distribution of the prime numbers, via Euler's 1737 Product Formula. In 1859, with tremendous insight, Bernhard Riemann extended  $\zeta$  to all complex numbers  $z = x + iy$ : the link became the Prime Number Theorem, which would remain unproved for a further forty years; even more, the error in the Theorem's estimate was related precisely to the distance of zeros of  $\zeta$  from the critical line  $x = 1/2$ : if, for  $x > 0$ ,  $\zeta(x + iy) = 0$  always implied that  $x = 1/2$  then this error was confirmed to be as small as possible. Thus Riemann was led to his monumental Hypothesis, known simply as RH: all zeros of  $\zeta(z)$  in the right half-plane lie on the the line  $z = 1/2 + iy$ .

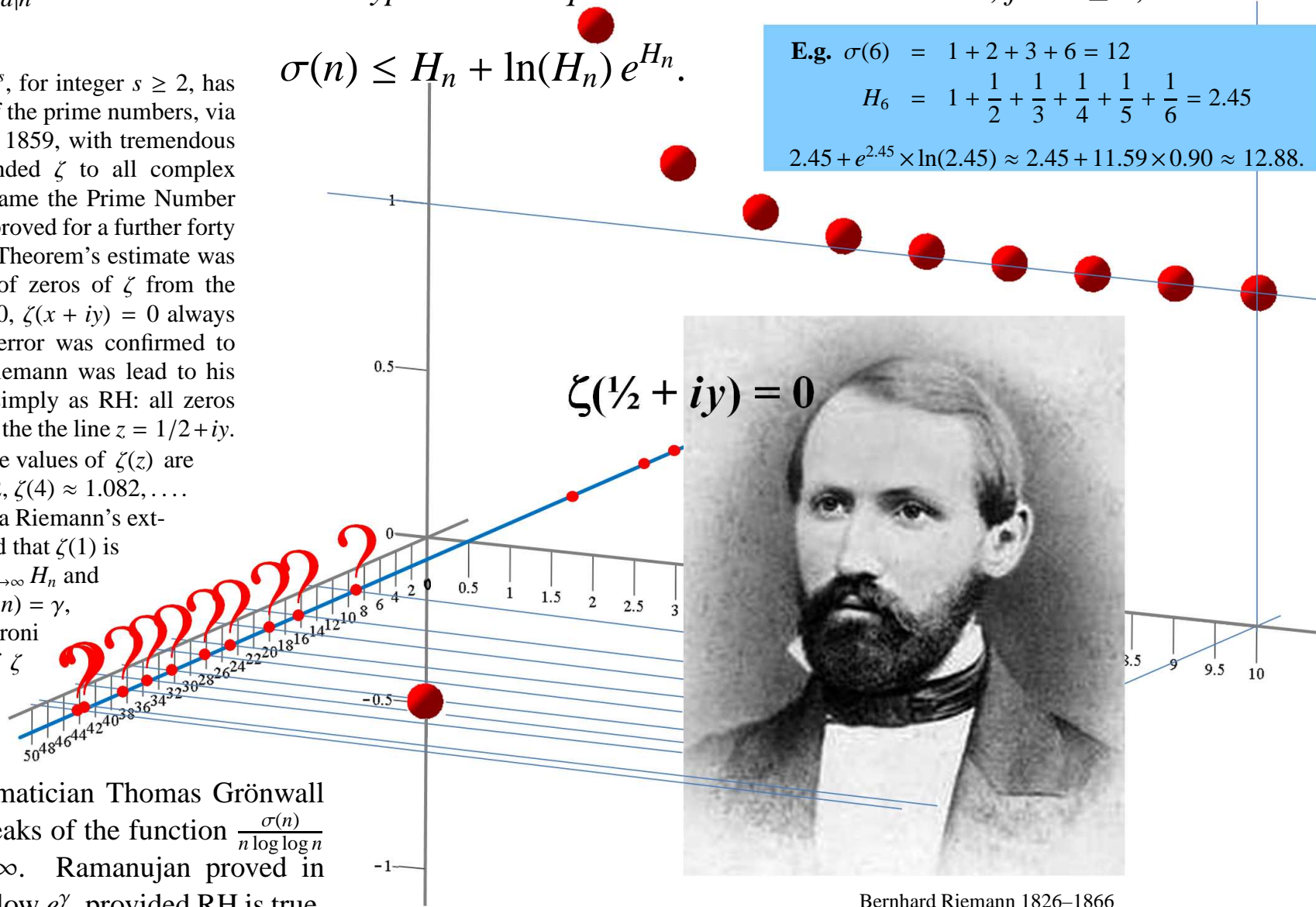
In the illustration on the right, some values of  $\zeta(z)$  are plotted:  $\zeta(2) \approx 1.645$ ,  $\zeta(3) \approx 1.202$ ,  $\zeta(4) \approx 1.082, \dots$

Note that  $\zeta(0)$  evaluates to  $-1/2$  via Riemann's extension to all complex numbers; and that  $\zeta(1)$  is not defined since  $\sum_{k=1}^{\infty} 1/k^1 = \lim_{n \rightarrow \infty} H_n$  and  $H_n > \ln n$ ; indeed,  $\lim_{n \rightarrow \infty} (H_n - \ln n) = \gamma$ ,

$\gamma \approx 0.577$  being the Euler–Mascheroni constant. The first ten zero pairs of  $\zeta$  are also plotted: the first being  $\approx \frac{1}{2} \pm 14.135i$ , and the tenth being  $\approx \frac{1}{2} \pm 49.774i$ .

In 1913 the Swedish mathematician Thomas Grönwall proved that the successive peaks of the function  $\frac{\sigma(n)}{n \log \log n}$  approach  $e^\gamma$  as  $n$  goes to  $\infty$ . Ramanujan proved in 1915 that these peaks stay below  $e^\gamma$ , provided RH is true.

In 1984 Guy Robin made the striking discovery that a reverse implication is true: if, for all  $n > 7!$ ,  $\sigma(n) < e^\gamma n \log \log n$  then RH follows. And in 2002 Jeffrey Lagarias eliminated the  $\gamma$  and strengthened the condition on  $n$  to give the truly 'elementary' equivalent to RH stated above.



Bernhard Riemann 1826–1866

Web link: [arxiv.org/abs/math/0008177](https://arxiv.org/abs/math/0008177)

Further reading: *Equivalents of the Riemann Hypothesis: Volume 1, Arithmetic Equivalents* by Kevin Broughan

Created by Robin Whitty for [www.theoremoftheday.org](http://www.theoremoftheday.org)

