THEOREM OF THE DAY

The Robin–Lagarias Theorem *Let H_n denote the n-th harmonic number $\sum_{i=1}^{n} \frac{1}{i}$, and let $\sigma(n)$ denote the divisor function $\sum_{d|n} d$. Then the Riemann Hypothesis is equivalent to the statement that, for $n \geq 1$,*

$$\sigma(n) \leq H_n + \ln(H_n)\, e^{H_n}.$$

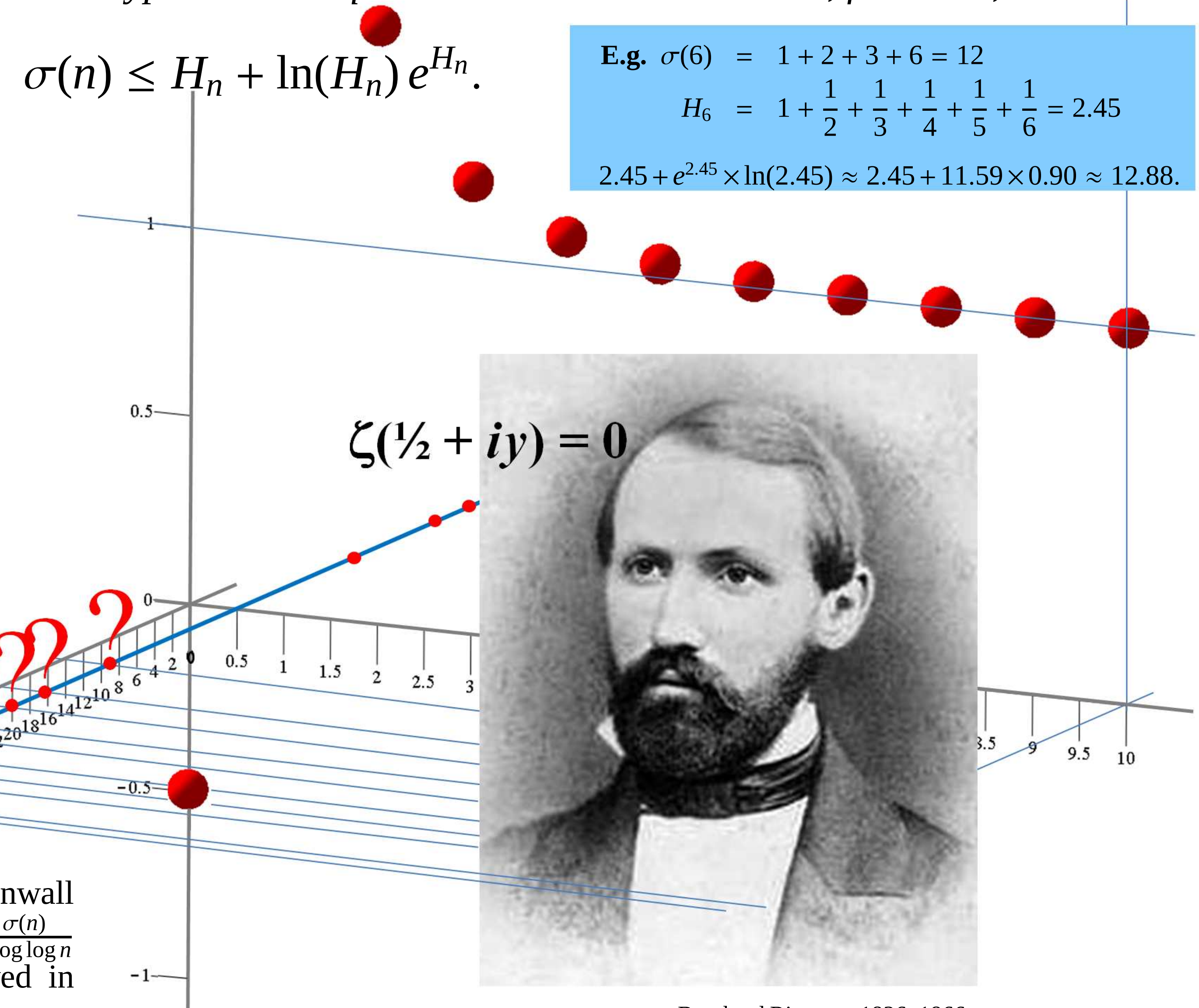
E.g.

$$\sigma(6) = 1+2+3+6 = 12$$

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45$$

$$2.45 + e^{2.45} \times \ln(2.45) \approx 2.45 + 11.59 \times 0.90 \approx 12.88.$$

The zeta function $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, for integer $s \geq 2$, has a close link with the distribution of the prime numbers, via Euler's 1737 Product Formula. In 1859, with tremendous insight, Bernhard Riemann extended ζ to all complex numbers $z = x + iy$: the link became the Prime Number Theorem, which would remain unproved for a further forty years; even more, the error in the Theorem's estimate was related precisely to the distance of zeros of ζ from the *critical line* $x = 1/2$: if, for $x > 0$, $\zeta(x + iy) = 0$ always implied that $x = 1/2$ then this error was confirmed to be as small as possible. Thus Riemann was lead to his monumental Hypothesis, known simply as RH: all zeros of $\zeta(z)$ in the right half-plane lie on the the line $z = 1/2 + iy$. In the illustration on the right, some values of $\zeta(z)$ are plotted: $\zeta(2) \approx 1.645$, $\zeta(3) \approx 1.202$, $\zeta(4) \approx 1.082, \ldots$. Note that $\zeta(0)$ evaluates to $-1/2$ via Riemann's extension to all complex numbers; and that $\zeta(1)$ is not defined since $\sum_{k=1}^{\infty} 1/k^1 = \lim_{n\to\infty} H_n$ and $H_n > \ln n$; indeed, $\lim_{n\to\infty}(H_n - \ln n) = \gamma$, $\gamma \approx 0.577$ being the Euler–Mascheroni constant. The first ten zero pairs of ζ are also plotted: the first being $\approx \frac{1}{2} \pm 14.135i$, and the tenth being $\approx \frac{1}{2} \pm 49.774i$.



Bernhard Riemann 1826–1866

In 1913 the Swedish mathematician Thomas Grönwall proved that the successive peaks of the function $\frac{\sigma(n)}{n \log\log n}$ approach e^{γ} as n goes to ∞. Ramanujan proved in 1915 that these peaks stay below e^{γ}, provided RH is true.

In 1984 Guy Robin made the striking discovery that a reverse implication is true: if, for all $n > 7!$, $\sigma(n) < e^{\gamma} n \log\log n$ then RH follows. And in 2002 Jeffrey Lagarias eliminated the γ and strengthened the condition on n to give the truly 'elementary' equivalent to RH stated above.

Web link: arxiv.org/abs/math/0008177

Further reading: *Equivalents of the Riemann Hypothesis: Volume 1, Arithmetic Equivalents* by Kevin Broughan