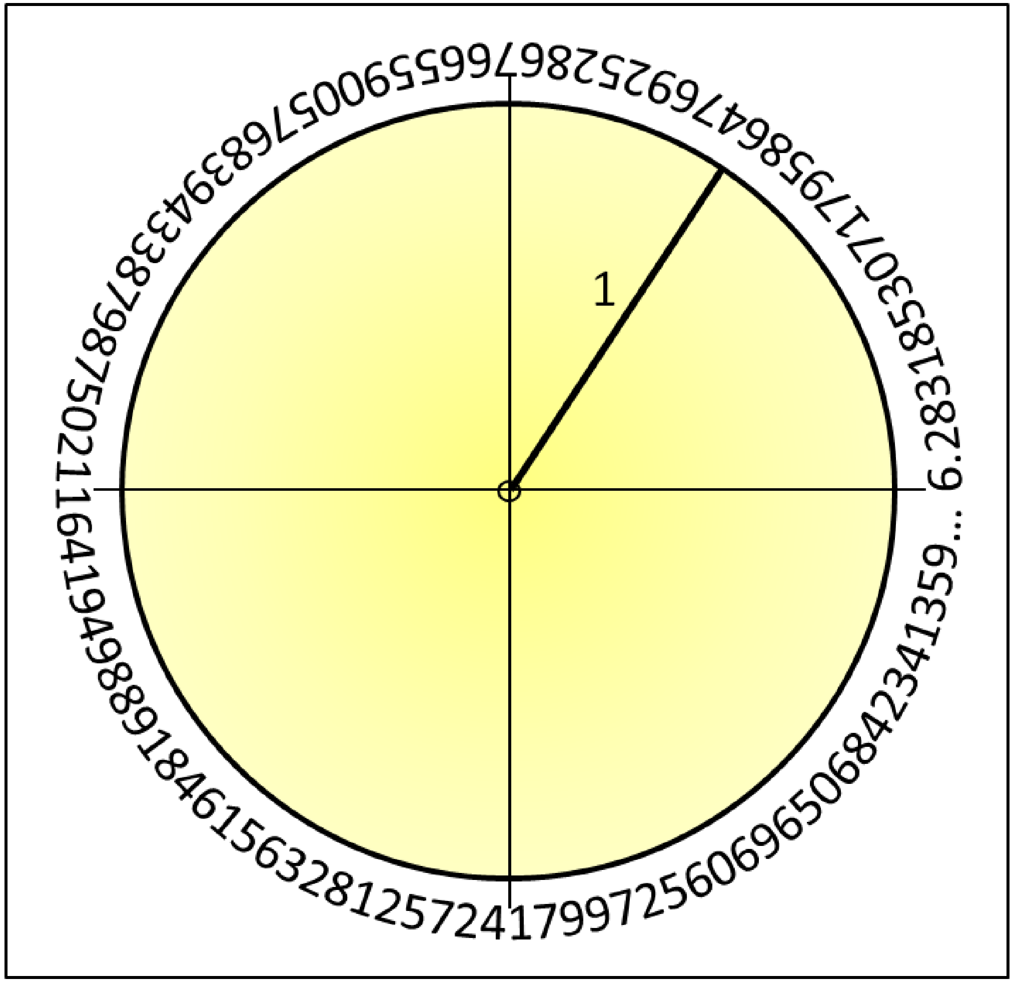
THEOREM OF THE DAY

Irrationality of Circumference of Unit Circle *The circumference, τ, of the unit circle is not rational, i.e. it cannot be expressed as a ratio of two integers.*



The unit circle is that described by a radius of 1, depicted here inclined at an angle of 1 radian ($\approx 57.3°$) to the horizontal axis. Since 1 radian subtends an arc length of 1 radius, it follows that $\tau = 6.283\ldots$ is the number of radians in a circle of any radius. The conversions between degrees and radians follow:

$$x \text{ degrees} = x \times \frac{\tau}{360} \text{ radians}, \qquad y \text{ radians} = y \times \frac{360}{\tau} \text{ degrees}.$$

The substitution of an irrational subdivision of the circle for a rational (indeed integer) subdivision is intimately connected with the development of the calculus: the slope of the sine curve at x is necessarily given by $\cos(x)$ only if x is measured in radians, etc. The first demonstration of the irrationality of the circle constant relied on this substitution: Johann Heinrich Lambert showed that if x, *measured in radians*, is rational then $\tan(x)$ cannot be rational. A textbook application of the contrapositive follows (notation: '$\in \mathbb{Q}$' means 'in the rationals'; '$\notin$' means 'not in'; '$\Rightarrow$' denotes logical implication):

$$\begin{array}{rrcl} & x \in \mathbb{Q} & \Rightarrow & \tan(x) \notin \mathbb{Q} \\ \text{logically equivalent to:} & \textbf{not } \tan(x) \notin \mathbb{Q} & \Rightarrow & \textbf{not } x \in \mathbb{Q} \\ \text{logically equivalent to:} & \tan(x) \in \mathbb{Q} & \Rightarrow & x \notin \mathbb{Q} \\ \text{therefore:} & \tan\left(\frac{\tau}{8}\right) = 1 \in \mathbb{Q} & \Rightarrow & \frac{\tau}{8} \notin \mathbb{Q}. \end{array}$$

How did Lambert prove the first step in this argument? By giving an infinite continued fraction for $\tan(x)$:

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \ddots}}}}}.$$

Assuming x is rational, such an infinite continued fraction may only converge to an irrational value.

The introduction of radian measure is credited to Roger Cotes (1682–1716) who may also deserve credit for explicit radian-based differentiation of trigonometric functions. Lambert proved his result in 1761. Charles Hermite published a proof using basic calculus in 1871 and this is the basis for a variety of short 'modern-day' proofs.

Web link: Cotes on $d \sin x / dx$: fredrickey.info/hm/CalcNotes/Sine-Deriv.pdf and on $e^{ix} = \cos x + i \sin x$: thonyc.wordpress.com/2013/07/10/. A lovely 1-page irrationality proof was published by Ivan Niven in 1947 here: www.ams.org/journals/bull/1947-53-06/.

Further reading: *Irrational Numbers* by Ivan Niven, MAA, reprint edition, 2005.