## THEOREM OF THE DAY

Irrationality of Circumference of Unit Circle The circumference, $\tau$, of the unit circle is not rational, i.e. it cannot be expressed as a ratio of two integers.


The introduction of radian measure is credited to Roger Cotes (1682-1716) who may also deserve credit for explicit radianbased differentiation of trigonometric functions. Lambert first announced his result in 1766. Charles Hermite published a proof using basic calculus in 1873 and this is the basis for a variety of short 'modern-day' proofs.

The unit circle is that described by a radius of 1 , depicted here inclined at an angle of 1 radian ( $\approx 57.3^{\circ}$ ) to the horizontal axis. Since 1 radian subtends an arc length of 1 radius, it follows that $\tau=6.283 \ldots$ is the number of radians in a circle of any radius. The conversions between degrees and radians follow:

$$
x \text { degrees }=x \times \frac{\tau}{360} \text { radians, } \quad y \text { radians }=y \times \frac{360}{\tau} \text { degrees }
$$

The substitution of an irrational subdivision of the circle for a rational (indeed integer) subdivision is intimately connected with the development of the calculus: the slope of the sine curve at $x$ is necessarily given by $\cos (x)$ only if $x$ is measured in radians, etc. The first demonstration of the irrationality of the circle constant relied on this substitution: Johann Heinrich Lambert showed that if $x$, measured in radians, is rational then $\tan (x)$ cannot be rational. A textbook application of the contrapositive follows (notation: ' $\in \mathbb{Q}$ ' means 'in the rationals’; ' $\notin$ 'means 'not in’; ' $\Rightarrow$ ' denotes logical implication):

$$
x \in \mathbb{Q} \Rightarrow \tan (x) \notin \mathbb{Q}
$$

logically equivalent to: not $\tan (x) \notin \mathbb{Q} \Rightarrow \operatorname{not} x \in \mathbb{Q}$
logically equivalent to: $\tan (x) \in \mathbb{Q} \Rightarrow x \notin \mathbb{Q}$

$$
\text { therefore: } \tan \left(\frac{\tau}{8}\right)=1 \in \mathbb{Q} \Rightarrow \frac{\tau}{8} \notin \mathbb{Q} \text {. }
$$

How did Lambert prove the first step in this argument? By giving an infinite continued fraction for $\tan (x)$ :

$$
\tan (x)=\frac{x}{1-\frac{x^{2}}{3-\frac{x^{2}}{5-\frac{x^{2}}{7-\frac{x^{2}}{9-\ddots}}}}}
$$

Assuming $x$ is rational, such an infinite continued fraction may only converge to an irrational value.

Web link: Cotes on $d \sin x / d x$ : fredrickey.info/hm/CalcNotes/Sine-Deriv.pdf and on $e^{i x}=\cos x+i \sin x$ : thonyc.wordpress.com/2013/07/10/. A lovely 1-page irrationality proof was published by Ivan Niven in 1947 here: www.ams.org/journals/bull/1947-53-06/.
Further reading: Irrational Numbers by Ivan Niven, MAA, reprint edition, 2005.
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