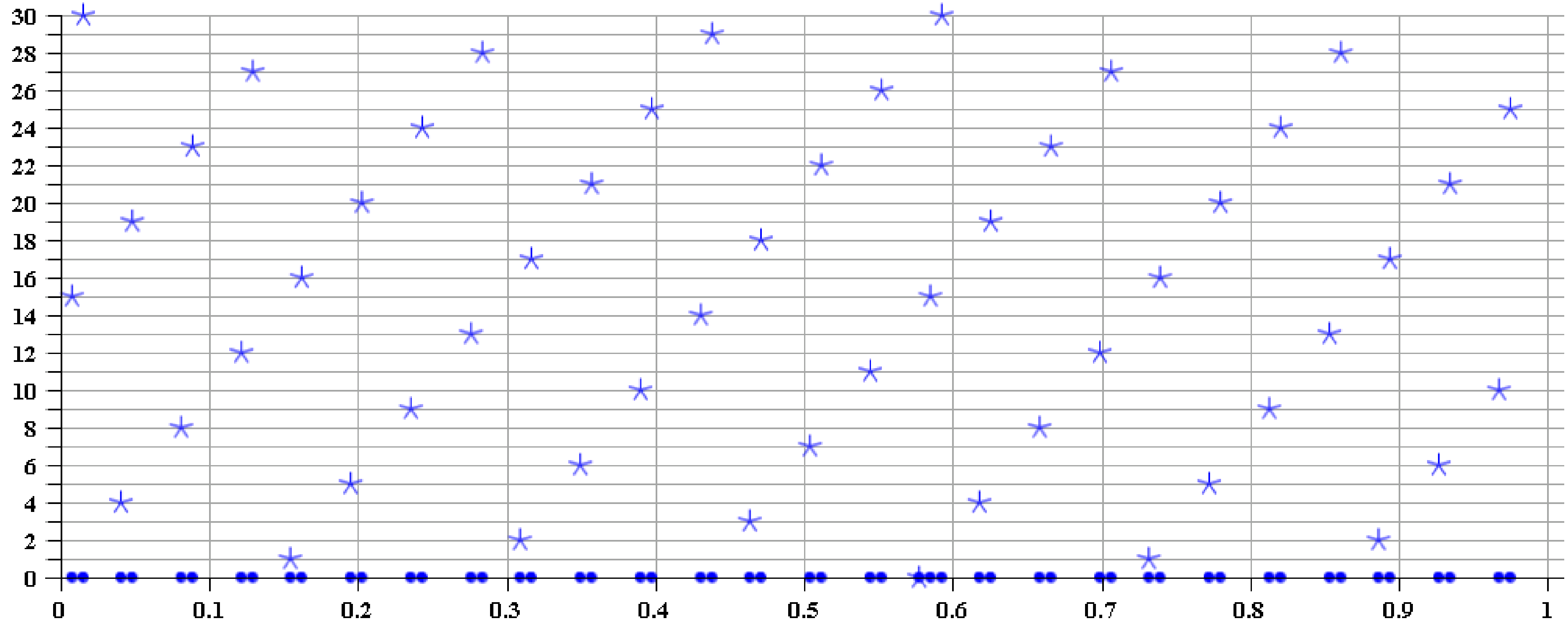
THEOREM OF THE DAY

The Three-Distance Theorem *Let $\alpha \in (0,1)$ be irrational and let N be a positive integer. Then the set of lengths $\{k\alpha \,|\, 0 \le k \le N\}$, measured around the unit-circumference circle, partitions the circle into $N+1$ intervals, whose lengths take just two values, or three values of which one is the sum of the other two.*



The theorem is applied in the above plot to the Euler–Mascheroni constant γ, the limiting value of $H_n - \ln n$ where H_n is the n-th harmonic number: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. To 10 decimal places this is 0.5772156649. Plotting the value of $k\gamma$ around the unit-circumference circle is equivalent to plotting its fractional part $k\gamma - [k\gamma]$ in the unit interval $(0,1)$ (with $[x]$ denoting the integer part of real number x). These values are displayed on the x-axis of the above plot for $k = 1, \ldots, N = 53$. The points $(k\gamma - [k\gamma], [k\gamma])$ have also been plotted (as 'snowflakes'): this allows us to track the progression of the sequence γ, 2γ, $3\gamma, \ldots$ up through the snowstorm. Supposing (this is not even known!) that γ is irrational, then no two snowflakes will ever land on exactly the same spot, yet however many snowflakes land they will always be separated by at most three distances. For $N = 53$ these three distances form the set $\{-45\gamma + 26, -19\gamma + 11, 26\gamma - 15\}$ (approximately 0.0253, 0.0329, 0.0076, respectively). By $N = 1000$, the smallest distance has reduced to $395\gamma - 228 \approx 0.000188$, providing us with a rational approximation to γ which is accurate to 5 decimal places: $395\gamma - 228 \approx 0$ so $\gamma \approx 228/395$.

This result belongs, as is hinted above, to the field of Diophantine approximation. It was conjectured by Hugo Steinhaus (1887–1972) and proved by Stanisław Świerczkowski in his 1956 PhD thesis and, independently, by Vera Sós in 1957.

Web link: www.theoremoftheday.org/Docs/3dAlessandriBerthe.pdf (fine survey by Pascal Alessandri and Valérie Berthé; see section 3).

Further reading: *An Invitation to Modern Number Theory* by S. J. Miller and R. Takloo-Bighash, Princeton University Press, 2006.