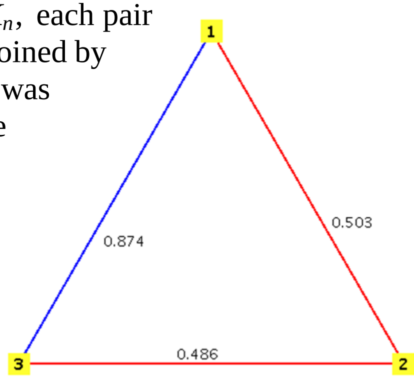
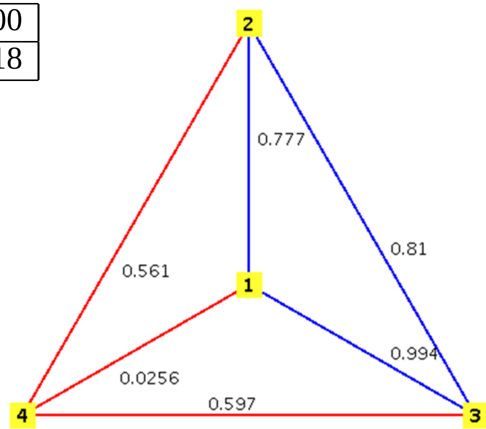
THEOREM OF THE DAY

Frieze's Theorem on Expected Minimum Tree Length *If the edges of the complete graph on n vertices are assigned weights independently uniformly at random from the interval* $[0, 1]$ *then the expected length of a minimum-weight spanning tree tends, as* $n \to \infty$, *to* $\zeta(3) \approx 1.20206$.

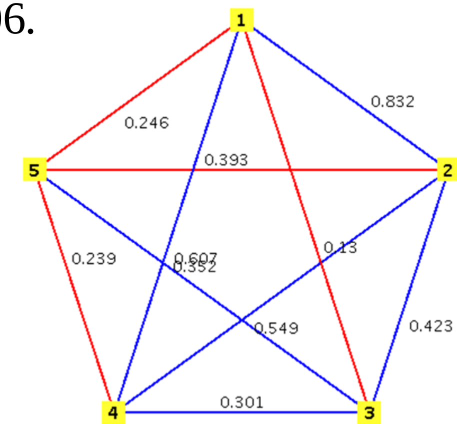
n	3	4	5	6	7	8	9	10	1000
$\overline{w}$	0.74	0.81	0.10	1.04	1.12	1.07	1.00	1.09	1.2018



$w = 0.989$



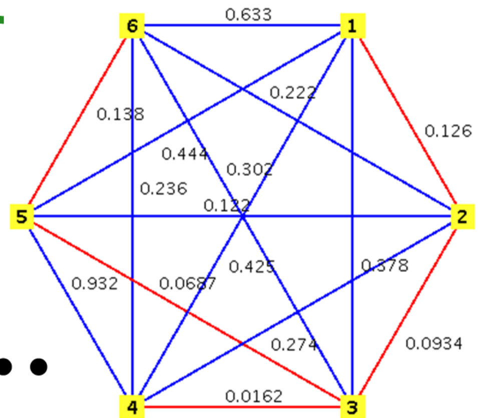
$w = 1.184$



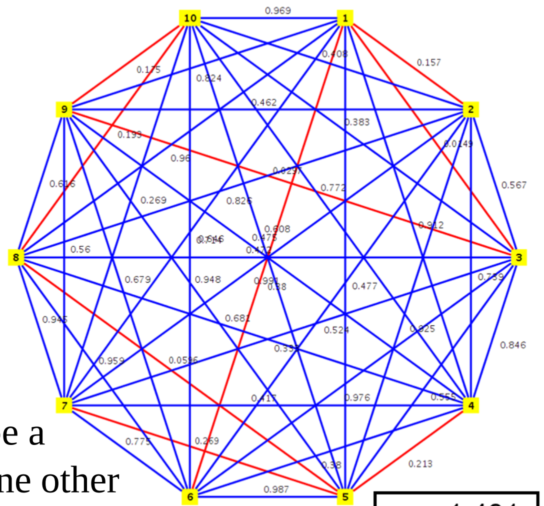
$w = 1.008$

In the *complete graph*, K_n, each pair from n vertices, $n \geq 1$, is joined by an edge. The above table was produced using the Maple GraphTheory package: for $3 \leq n \leq 10$, 25 copies of K_n were generated, with edges given random weights from the interval $[0, 1]$; and the mean length $\overline{w}$ calculated of a minimum spanning tree (a subset of edges connecting all vertices for the least possible total edge weight; individual examples for $n = 3, 4, 5, 6$ and 10 are shown on the right). The experiment was repeated with 25 copies of K_{1000}: the mean value minimum spanning tree length approximated $\zeta(3)$ to 3 decimal places.

$$\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + 1/5^3 + \cdots$$



$w = 0.442$



$w = 1.491$

An n-vertex spanning tree is a subset of $n-1$ edges; an arbitrary such subset in our weighted K_n will have expected total weight $(n-1) \times \frac{1}{2}$; so it is not even obvious that minimum spanning tree length should remain bounded as $n \to \infty$, let alone that its expected value, as discovered by Alan Frieze in 1985, should be a constant as intriguing as $\zeta(3)$ (whose reciprocal, to mention just one other property, is the proportion, as $n \to \infty$, of coprime triples in the set $\{1, \ldots, n\}$).

Web link: www.johndcook.com/blog/2017/08/09/

Further reading: *The Probabilistic Method, 3rd ed.* by Noga Alon and Joel H. Spencer, WileyBlackwell, 2008.