**THEOREM OF THE DAY**

**Frieze’s Theorem on Expected Minimum Tree Length** If the edges of the complete graph on \( n \) vertices are assigned weights independently uniformly at random from the interval \([0, 1]\) then the expected length of a minimum-weight spanning tree tends, as \( n \to \infty \), to \( \zeta(3) \approx 1.20206 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{w} )</td>
<td>0.74</td>
<td>0.81</td>
<td>1.00</td>
<td>1.04</td>
<td>1.12</td>
<td>1.07</td>
<td>1.00</td>
<td>1.09</td>
<td>1.2018</td>
</tr>
</tbody>
</table>

In the complete graph, \( K_n \), each pair from \( n \) vertices, \( n \geq 1 \), is joined by an edge. The above table was produced using the Maple `GraphTheory` package: for \( 3 \leq n \leq 10 \), 25 copies of \( K_n \) were generated, with edges given random weights from the interval \([0, 1]\); and the mean length \( \bar{w} \) calculated of a minimum spanning tree (a subset of edges connecting all vertices for the least possible total edge weight; individual examples for \( n = 3, 4, 5, 6 \) and 10 are shown on the right). The experiment was repeated with 25 copies of \( K_{1000} \); the mean value minimum spanning tree length approximated \( \zeta(3) \) to 3 decimal places.

An \( n \)-vertex spanning tree is a subset of \( n - 1 \) edges; an arbitrary such subset in our weighted \( K_n \) will have expected total weight \((n - 1) \times \frac{1}{3} \); so it is not even obvious that minimum spanning tree length should remain bounded as \( n \to \infty \), let alone that its expected value, as discovered by Alan Frieze in 1985, should be a constant as intriguing as \( \zeta(3) \) (whose reciprocal, to mention just one other property, is the proportion, as \( n \to \infty \), of coprime triples in the set \( \{1, \ldots, n\} \)).

\[
\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots
\]
