



# THEOREM OF THE DAY

**Von Neumann's Minimax Theorem** For any finite, two-player, zero-sum game the maximum value of the minimum expected gain for one player is equal to the minimum value of the maximum expected loss for the other; moreover each player has a mixed strategy which realises this equality.

Alice and Bob's game matrix:

$$\begin{matrix} & \text{red} & \text{blue} \\ \text{red} & (r+b & -2r) \\ \text{blue} & (-2b & r+b) \end{matrix} \leftarrow$$

↓ Add  $2(r+b)$  to each entry to make positive:

$$G^+ = \begin{pmatrix} 3(r+b) & 2b \\ 2r & 3(r+b) \end{pmatrix}$$

The associated linear programme solved:

basis	eqn	z	$x_1$	$x_2$	$x_3$	$x_4$	RS	
	0	1	0	0	$\frac{r+3b}{X}$	$\frac{3r+b}{X}$	$\frac{4(r+b)}{X}$	$\frac{q_i}{x_i v_B}$
$x_1$	1	0	1	0	$\frac{3(r+b)}{X}$	$\frac{-2b}{X}$	$\frac{3r+b}{X}$	$\frac{3r+b}{4(r+b)}$
$x_2$	2	0	0	1	$\frac{-2r}{X}$	$\frac{3(r+b)}{X}$	$\frac{r+3b}{X}$	$\frac{r+3b}{4(r+b)}$

$$X = 9r^2 + 9b^2 + 14rb$$

## Rules:

1. Alice and Bob agree a 'red' price  $r$  and a 'blue' price  $b$ .
2. Both players choose a colour.
3. If Alice and Bob chose the same colour then Bob pays Alice both prices; if they choose differently then Alice pays Bob twice her chosen colour price.



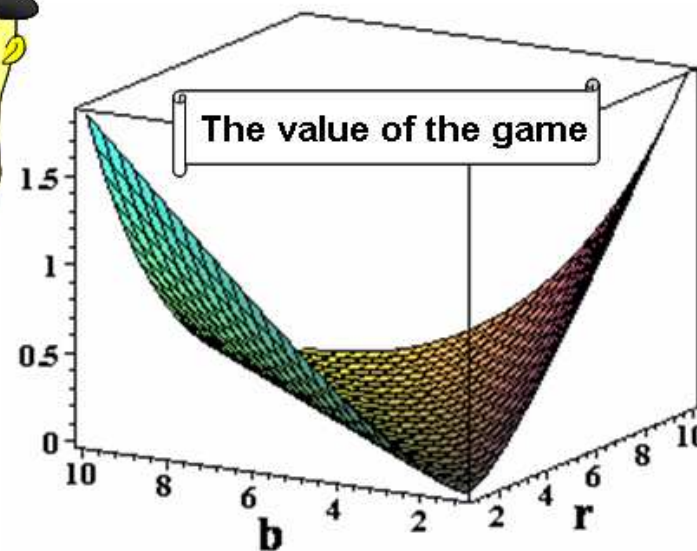
	A	B	red	blue
red			13	-16
blue			-10	13



**Finite:** each player has a finite list of strategies to choose from;

**Mixed strategy:** a rule which chooses each strategy with a certain probability;

**Zero-sum:** any gain by one player corresponds to an equal loss to the other.



This is a lovely application of linear programming duality.  $A$ 's strategies are 'choose red' and 'choose blue'; suppose she attaches probabilities  $p_1$  and  $p_2$  to these choices, respectively, with  $p_1 + p_2 = 1$ : this is her mixed strategy. Suppose  $B$  chooses red and blue with probabilities  $q_1$  and  $q_2$ , respectively, with  $q_1 + q_2 = 1$ . In the positive version of the game, represented above left by  $G^+$ , suppose that  $B$ 's maximum expected loss is  $v_B$ . Then  $B$  is trying to minimise  $v_B$  subject to  $3(r+b)q_1 + 2bq_2 \leq v_B$  and  $2rq_1 + 3(r+b)q_2 \leq v_B$ . Divide through by  $v_B$ : since we made our game positive,  $v_B$  must be positive and this will preserve the inequalities. Letting  $x_i = q_i/v_B$  we have  $3(r+b)x_1 + 2bx_2 \leq 1$  and  $2rx_1 + 3(r+b)x_2 \leq 1$ . Meanwhile,  $x_1 + x_2 = (q_1 + q_2)/v_B = 1/v_B$ , so  $B$  minimises  $v_B$  by maximising  $x_1 + x_2$ ; and hey presto! we have a standard linear programme. This is solved above left, and by duality  $B$ 's minimum maximum expected loss equals  $A$ 's maximum minimum expected gain and is given by the reciprocal of the top-right value in the simplex tableau:  $X/4(r+b)$ . For the original game we must subtract  $2(r+b)$  giving the value of the original game as  $(r-b)^2/4(r+b)$ . This is plotted above right: we see that Alice never loses and Bob breaks even only if the red and blue prices are equal.

John von Neumann's theorem appears in a classic 1928 paper in which he single-handedly invented Game Theory.

**Web link:** [math.ucr.edu/home/baez/games/games\\_1.html](http://math.ucr.edu/home/baez/games/games_1.html) (von Neumann's theorem appears in lecture 20).

**Further reading:** *Game Theory: Mathematical Models of Conflict* by A.J. Jones, Woodhead Publishing, 2000.

