THEOREM OF THE DAY

Von Neumann's Minimax Theorem *For any finite, two-player, zero-sum game the maximum value of the minimum expected gain for one player is equal to the minimum value of the maximum expected loss for the other; moreover each player has a mixed strategy which realises this equality.*

Alice and Bob's **game matrix**:

$$\begin{array}{c} \\ \text{red} \\ \text{blue} \end{array} \begin{array}{c} \begin{array}{cc} \text{red} & \text{blue} \end{array} \\ \begin{pmatrix} r+b & -2r \\ -2b & r+b \end{pmatrix} \end{array}$$

Add $2(r+b)$ to each entry to make positive:

$$G^+ = \begin{pmatrix} 3(r+b) & 2b \\ 2r & 3(r+b) \end{pmatrix}$$

The associated linear programme solved:

basis	eqn	z	x_1	x_2	x_3	x_4	RS	$q_i = x_i v_B$
			$p_i =$		$\frac{r+3b}{4(r+b)}$	$\frac{3r+b}{4(r+b)}$		
	0	1	0	0	$\frac{r+3b}{X}$	$\frac{3r+b}{X}$	$\frac{4(r+b)}{X}$	
x_1	1	0	1	0	$\frac{3(r+b)}{X}$	$\frac{-2b}{X}$	$\frac{3r+b}{X}$	$\frac{3r+b}{4(r+b)}$
x_2	2	0	0	1	$\frac{-2r}{X}$	$\frac{3(r+b)}{X}$	$\frac{r+3b}{X}$	$\frac{r+3b}{4(r+b)}$

$X = 9r^2 + 9b^2 + 14rb$

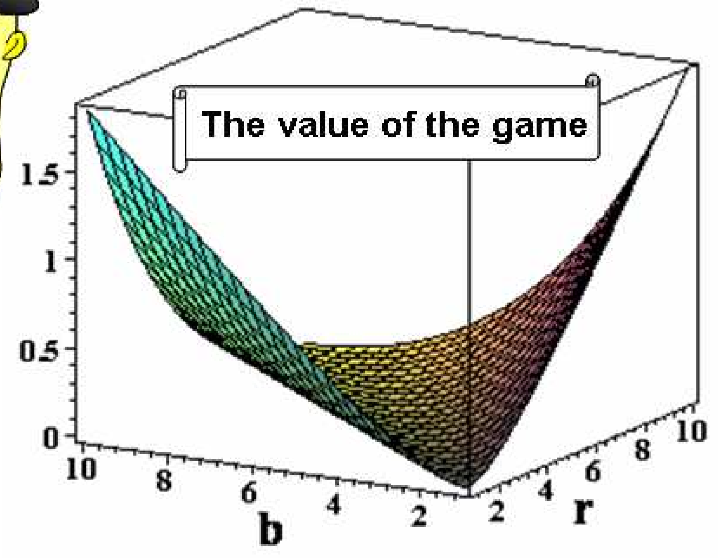
Rules:

1. Alice and Bob agree a 'red' price r and a 'blue' price b.
2. Both players choose a colour.
3. If Alice and Bob chose the same colour then Bob pays Alice both prices; if they choose differently then Alice pays Bob twice her chosen colour price.

A \ B	red	blue
red	13	–16
blue	–10	13

€8 €5

Finite: each player has a finite list of strategies to choose from;
Mixed strategy: a rule which chooses each strategy with a certain probability;
Zero-sum: any gain by one player corresponds to an equal loss to the other.



This is a lovely application of linear programming duality. A's strategies are 'choose red' and 'choose blue'; suppose she attaches probabilities p_1 and p_2 to these choices, respectively, with $p_1 + p_2 = 1$: this is her mixed strategy. Suppose B chooses red and blue with probabilities q_1 and q_2, respectively, with $q_1 + q_2 = 1$. In the positive version of the game, represented above left by G^+, suppose that B's maximum expected loss is v_B. Then B is trying minimise v_B subject to $3(r+b)q_1 + 2bq_2 \leq v_B$ and $2rq_1 + 3(r+b)q_2 \leq v_B$. Divide through by v_B: since we made our game positive, v_B must be positive and this will preserve the inequalities. Letting $x_i = q_i/v_B$ we have $3(r+b)x_1 + 2bx_2 \leq 1$ and $2rx_1 + 3(r+b)x_2 \leq 1$. Meanwhile, $x_1 + x_2 = (q_1 + q_2)/v_B = 1/v_B$, so B *minimises* v_B by *maximising* $x_1 + x_2$; and hey presto! we have a standard linear programme. This is solved above left, and by duality B's minimum maximum expected loss equals A's maximum minimum expected gain and is given by the reciprocal of the top-right value in the simplex tableau: $X/4(r+b)$. For the original game we must subtract $2(r+b)$ giving the *value* of the original game as $(r-b)^2/4(r+b)$. This is plotted above right: we see that Alice never loses and Bob breaks even only if the red and blue prices are equal.

John von Neumann's theorem appears in a classic 1928 paper in which he single-handedly invented Game Theory.

Web link: math.ucr.edu/home/baez/games/games_1.html (von Neumann's theorem appears in lecture 20).
Further reading: *Game Theory: Mathematical Models of Conflict* by A.J. Jones, Woodhead Publishing, 2000.