Rewley House June 2013 **Tau vs. Pi**

A Non-Regular Continued Fraction for τ

In 1998 Jerry Lange gave a beautiful new continued fraction for π:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ldots}}}.$$

A very short elegant proof was subsequently found by Douglas Bowman and Lange published this with two other proofs in 1999. From a general theorem of Leonhard Euler we derive:

$$\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \ldots + \frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cfrac{a_3^2}{\ldots + \cfrac{a_{k-1}^2}{a_k - a_{k-1}}}}}}, \quad \text{e.g.} \quad \frac{1}{a_1} - \frac{1}{a_2} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1}}, \text{etc.}$$

An infinite summation (due to the 15th century Indian scholar Kerala Gargya Nilakantha) gives:

$$\frac{\pi - 3}{4} = \frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)}. \qquad (1)$$

Bowman applies Euler's transformation, using $a_k = 2k(2k+1)(2k+2)$, to produce Lange's continued fraction. You can check that $a_k - a_{k-1} = 24k^2 = 6.(2k)^2$ for $k \geq 2$, and $a_1 = 6.2^2$. So:

$$\frac{\pi - 3}{4} = \cfrac{1}{6.4 + \cfrac{2^2.3^2.4^2}{6.4^2 + \cfrac{4^2.5^2.6^2}{6.6^2 + \cfrac{6^2.7^2.8^2}{6.8^2 + \ldots}}}} \quad \text{or} \quad \pi = 3 + \cfrac{\cancel{4}}{6.\cancel{4} + \cfrac{\cancel{2^2}.3^2.\cancel{4^2}}{6.\cancel{4^2} + \cfrac{\cancel{4^2}.5^2.\cancel{6^2}}{6.\cancel{6^2} + \cfrac{\cancel{6^2}.7^2.\cancel{8^2}}{6.\cancel{8^2} + \ldots}}}} = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ldots}}}.$$

Of course we can simply double both sides of Lange's continued fraction to get a non-regular continued fraction for τ but this leaves a 2 in the first numerator and makes the result inelegant. The appropriate course of action would seem to be to double both sides of equation (1):

$$\frac{\tau - 6}{4} = 2\left(\frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \ldots\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(2k+1)(2k+2)}. \qquad (2)$$

Now put $b_k = k(2k+1)(2k+2)$, with $b_k - b_{k-1} = 12k^2$ for $k \geq 2$, and $b_1 = 12$. So:

$$\frac{\tau - 6}{4} = \cfrac{1}{12 + \cfrac{1^2.3^2.4^2}{12.2^2 + \cfrac{2^2.5^2.6^2}{12.3^2 + \cfrac{3^2.7^2.8^2}{12.4^2 + \ldots}}}} \quad \text{or} \quad \tau = 6 + \cfrac{4}{12 + \cfrac{1^2.3^2.2^2.\cancel{2^2}}{12.\cancel{2^2} + \cfrac{\cancel{2^2}.5^2.2^2.\cancel{3^2}}{12.\cancel{3^2} + \cfrac{\cancel{3^2}.7^2.2^2.\cancel{4^2}}{12.\cancel{4^2} + \ldots}}}} = 6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12 + \ldots}}}.$$

Et voila—everything has doubled!

References

L.J.Lange, "An elegant continued fraction for π", *American Mathematical Monthly*, 106 (5) 1999, 456–458.
R. Roy, "The discovery of the series formula for π by Leibniz, Gregory and Nilakantha", *Mathematics Magazine*, 63 (5) 1990, 291–306.

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