THEOREM OF THE DAY NUMBER 156

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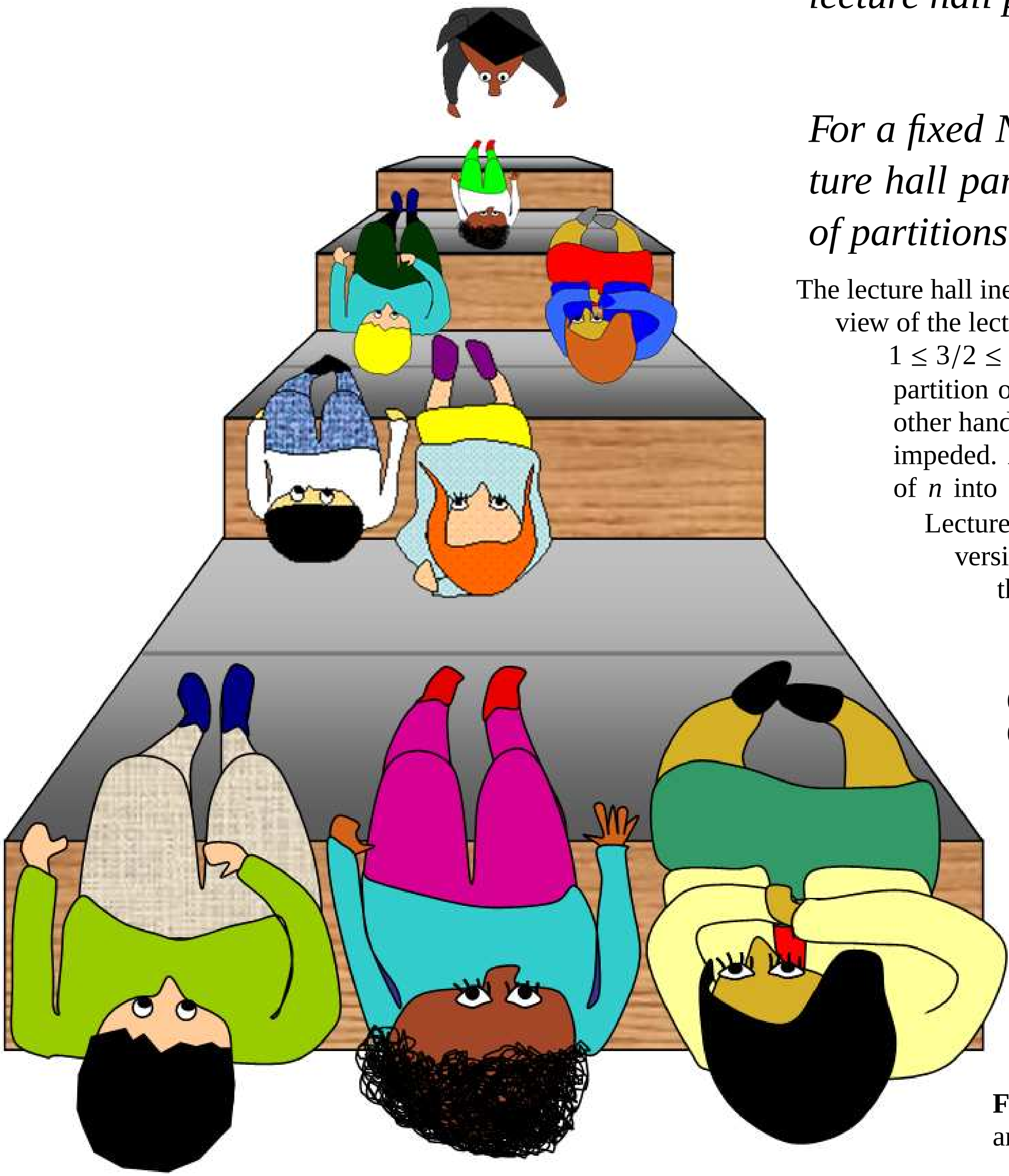
The Lecture Hall Partition Theorem *A list $(h_1, h_2, \ldots, h_N)$ of non-negative integers is said to be a lecture hall partition if*

$$0 \le \frac{h_1}{1} \le \frac{h_2}{2} \le \ldots \le \frac{h_N}{N}.$$

For a fixed N (the length of the lecture hall), the number of lecture hall partitions of any positive integer n equals the number of partitions of n into odd parts smaller than $2N$.



The lecture hall inequalities ensure that each row in a tiered lecture hall will have a clear view of the lecturer. The hall shown on the left has rows at height 1, 3, 5 and 7; since $1 \le 3/2 \le 5/3 \le 7/4$, the partition of 16 given by $1+3+5+7$ is a lecture hall partition of length $N = 4$. The partition of 16 given by $1 + 4 + 5 + 6$, on the other hand, represents a lecture hall in which the third and fourth row views are impeded. A famous partition identity of Leonhard Euler says that the partitions of n into distinct parts are equinumerous with those into odd parts; because Lecture Hall Partitions permit no equal (non-zero) parts, today's theorem is a version of Euler's identity with restricted maximum odd parts. If we allow the lecture hall to become infinitely long (removing the limit on N) we recover Euler's identity in the limit. Here are the 21 lecture halls of length $N = 4$ partitioning $n = 16$ (note, that zeros are allowed):

0, 0, 0, 16	0, 0, 1, 15	0, 0, 2, 14	0, 0, 3, 13	0, 1, 2, 13	0, 0, 4, 12	0, 1, 3, 12
0, 0, 5, 11	0, 1, 4, 11	0, 2, 3, 11	0, 0, 6, 10	0, 1, 5, 10	0, 2, 4, 10	1, 2, 3, 10
0, 1, 6, 9	0, 2, 5, 9	1, 2, 4, 9	0, 2, 6, 8	0, 3, 5, 8	1, 2, 5, 8	1, 3, 5, 7

The idea of a lecture hall partition arose unexpectedly in Kimmo Eriksson's work on Coxeter groups. He and Mireille Bousquet-Mélou proved the above (apparently difficult) identity and several deep generalisations in two influential papers in the late 1990s.

Web link: www-igm.univ-mlv.fr/~fpsac/FPSAC08/fpsac08.html: the 'List of Presentations' has a fine (but 3MB) talk by Carla Savage. (There is also a talk by Bousquet-Mélou although not on partitions).

Further reading: *Integer Partitions, 2nd revised ed.* by George E. Andrews and Kimmo Eriksson, Cambridge University Press, 2004, chapter 9.