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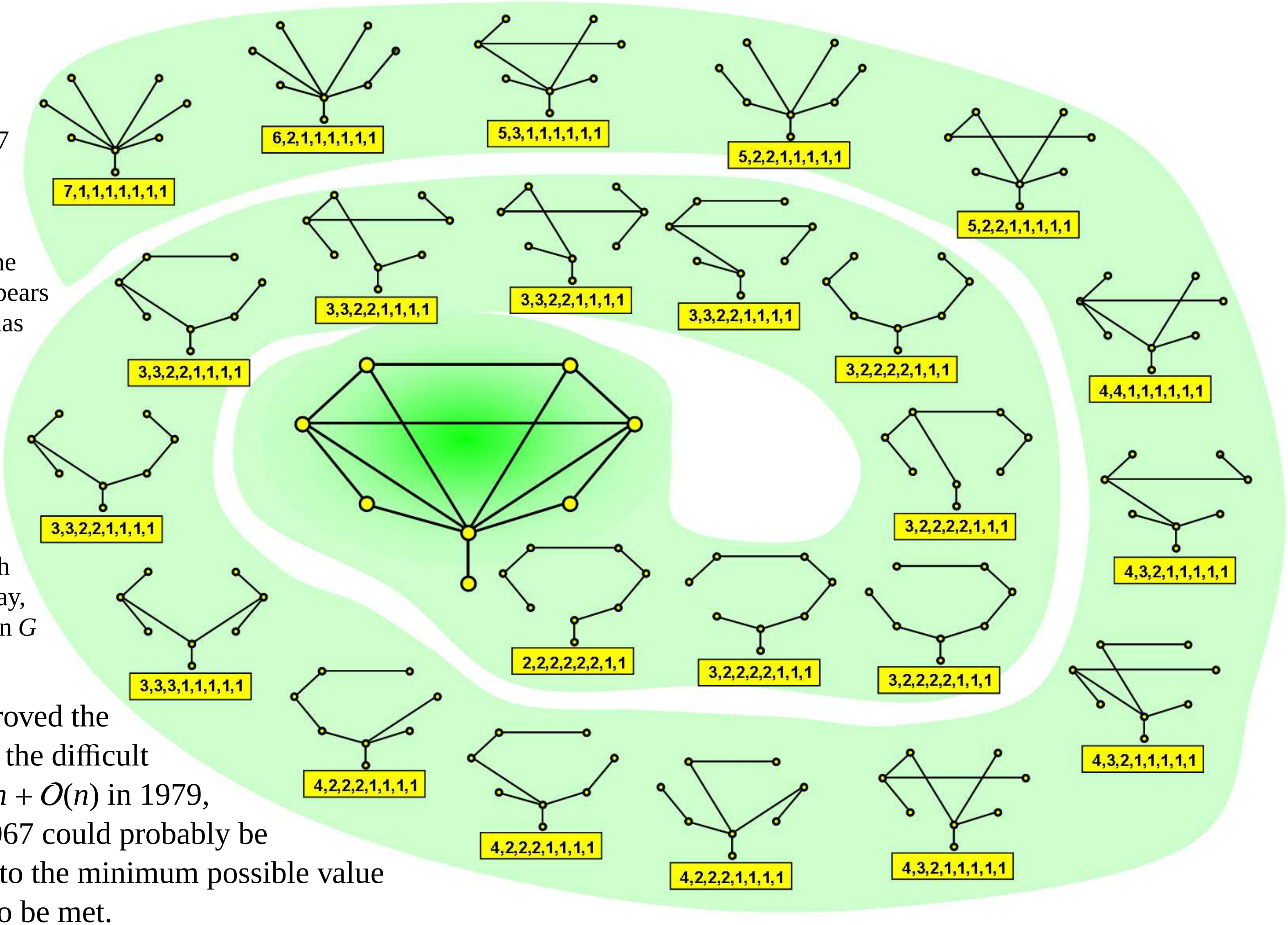
THEOREM OF THE DAY NUMBER 146

The Panarboreal Formula *Let $\mathcal{T}_n$ denote the set of all unlabelled trees on n edges and denote by $s(\mathcal{T}_n)$ the minimum number of edges which an $(n+1)$-vertex graph must have in order that it contains every tree in $\mathcal{T}_n$ as a subgraph. Then $s(\mathcal{T}_n) \sim cn \log n$ where c is a constant satisfying $1/2 \leq c \leq 5/\log 4$.*

There are 23 unlabelled trees having 7 edges; they are shown on the right, lexicographically ordered by degree sequence, together with an 8-vertex graph with 13 edges in which each one may be found as a subgraph. This appears to be the largest n for which $s(\mathcal{T}_n)$ has been calculated exactly; but it is easy to establish that $s(\mathcal{T}_n) \geq (1/2)n \log n$. For, given k, $1 \leq k \leq n+1$ we may always choose a tree in whose degree sequence the k-th entry $\geq n/k$. But now the same must hold for the degree sequence of any graph G containing this tree. So if G contains each tree in $\mathcal{T}_n$ and has degree sequence, say, $(d_1, \ldots, d_{n+1})$, then number of edges in G $= \frac{1}{2}\sum_{k=1}^{n+1} d_k \geq \frac{1}{2}\sum_{k=1}^{n+1} n/k > \frac{1}{2} n \log n$.



Fan Chung and Ron Graham proved the easy lower bound on $s(\mathcal{T}_n)$ and the difficult upper bound of $(5/\log 4)n \log n + \mathcal{O}(n)$ in 1979, mentioning that $5/\log 4 \approx 3.6067$ could probably be improved, possibly even down to the minimum possible value of 1/2. This challenge has yet to be met.

Web link: math.ucsd.edu/~fan/ (where all Chung's papers may be found in pdf format)

Further reading: *Erdős on Graphs: His Legacy of Unsolved Problems*, by Fan Chung and Ronald Graham, AK Peters, 1998, section 3.5.1.