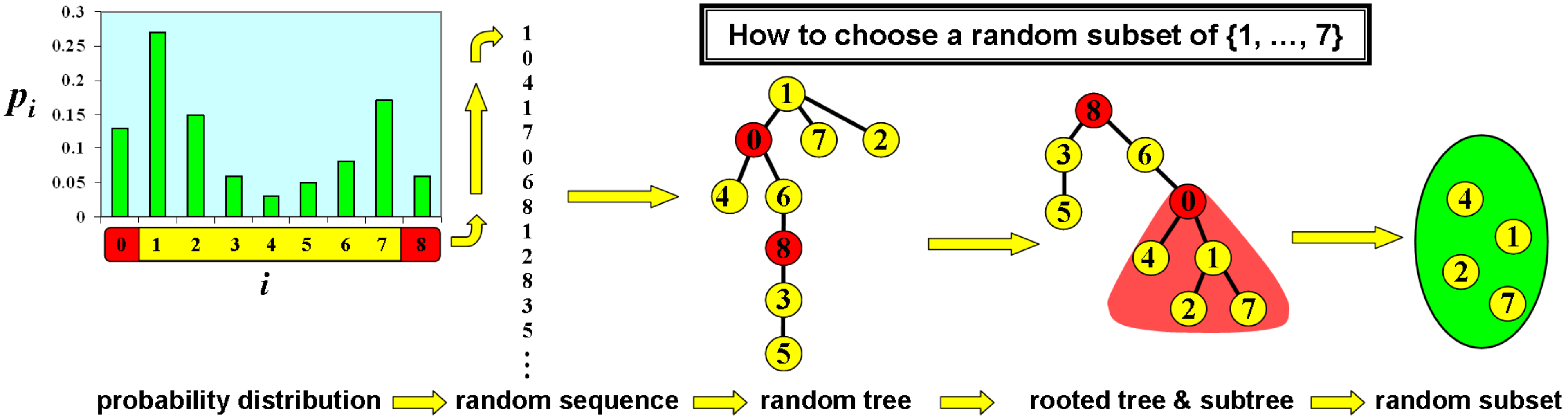
THEOREM OF THE DAY

The Abel-Hurwitz Binomial Theorem *For a positive integer n, let $[n]$ denote the set $\{1,\ldots,n\}$ and let $x, y, z_i, i \in [n]$, be $n+2$ variables. For $S \subseteq [n]$, let z_S denote the sum $\sum_{s\in S} z_s$. Then, summing over subsets A of $[n]$:*

$$\sum_{A\subseteq[n]} x(x+z_A)^{|A|-1}(y+z_{\bar{A}})^{|\bar{A}|} = (x+y+z_{[n]})^n. \qquad (1)$$



The great German mathematician Adolf Hurwitz published equation (1) in 1902 as a generalisation of Abel's generalisation of the binomial theorem. Abel's version is recovered by setting $x = X$, $y + z_{[n]} = Y$ and $z_1 = \ldots = z_n = Z$, for new variables X, Y and Z. We get an identity in three variables: $\sum_{k=0}^{n}\binom{n}{k}X(X-kZ)^{k-1}(Y+kZ)^{n-k} = (X+Y)^n$, with all terms in Z on the left magically cancelling each other. We further recover the classical binomial theorem by setting $Z = 0$. (If, on the other hand, we set $Z = -1$ and introduce two more variables, $Y = W + n$ and $n - k = K$, then we get $\sum_{K=0}^{n}\binom{n}{K}(X+n-K)^{n-K-1}(W+K)^K = X^{-1}(X+W+n)^n$: Abel's binomial theorem in the version preferred by the Wikipedia entry.)

During the 1990s, Jim Pitman applied another substitution for equation (1): $x = p_0, y = p_{n+1}$ and $z_i = p_i$, for $i = 1,\ldots,n$, where the p_i are a discrete probability distribution on $[n]\cup\{0, n+1\}$. Now the right-hand-side becomes $(p_0 + p_1 + \ldots + p_{n+1})^n$ in which each term is a probability of picking, with replacement, a certain subset of numbers in the range $0 \ldots n+1$. The left-hand-side of (1) now defines the distribution, over subsets A of $[n]$, of the probability, $\mathbb{P}(V(n) = A) = p_0(p_0 + p_A)^{|A|-1}(p_{n+1} + p_{\bar{A}})^{|\bar{A}|}$, that a random subset $V(n)$ will occur. This interpretation depends on making the right definition of 'random subset'; the picture above specifies one way of doing this.

Niels Henrik Abel published his binomial theorem in 1826 in the first issue of August Leopold Crelle's eponymous journal. The same issue saw the first full publication of Abel's stunning resolution in the negative of the question of whether quintic equations could be solved in radicals. Three years later, at the tragically young age of 26, having returned to his native Norway, Abel died of tuberculosis contracted in Paris.

Web link: statistics.berkeley.edu/tech-reports/500

Further reading: *Probability* by Jim Pitman, Springer, New York, 1999.