Bailey’s Theorem on Latin Squares

Let \( G \) be a finite group and let \( a \) and \( b \) be ordered listings of the elements of \( G \). Then the Cayley table \( L(a, b) \) is a quasi-complete Latin square if and only if \( a^{-1} \) and \( b \) are terraces for \( G \).

Given a set of \( n \) treatments to test, an \( n \times n \) Latin square is a grid in which each treatment appears exactly once in each row and column. In treatment testing, of a crop spray, for example, this factors out the influence of north-south and east-west geographical positioning (with respect to other fields, rivers etc). The Latin square constructed here is quasi-complete meaning that any two treatments appear adjacent to each other exactly twice, east to west, and exactly twice north to south. (Complete means every ordered pair occurs exactly once horizontally and once vertically, not true in the example above left.)

An ordered listing \( a = (a_1, a_2, \ldots, a_n) \) of the elements of a group \( G \) is a terrace if every set \( \{a_i, a_i^{-1}\} \) is represented twice in the sequence \( a^* = (1, a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-1}^{-1}a_n) \). The inverse \( a^{-1} \) denotes the listing \( (1, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}) \). For a square, let \( r \) denote its rotation through \( \tau/4 \) (a quarter circle) and \( f \) an east-west flip. The complete set of eight symmetries of the square constitutes the dihedral group \( D_8 \), with terraces \( a = (1, r, f, r^2f, rf, r^3f, r^2f, r^3r^2f) \) and \( b = (1, f, r, r^2, rf, r^3f, r^2f, r^3) \). The Cayley table, above right, multiples \( b \) (column headings) with \( a^{-1} \) (row headings). Both \( (a^{-1})^{-1} = a \) and \( b \) are terraces, so the theorem tells us that \( L(a^{-1}, b) \) is a quasi-complete Latin square.

Not all Latin squares are group multiplication tables but R.A. Bailey, in a classic 1984 paper, argued that, for reasonable numbers of treatments, group tables were sufficiently prevalent to offer a practical source of experimental designs. She was then able to use the machinery of group theory to bring the issue of quasi-complete Latin square construction to a ‘quasi-complete’ resolution (her conjecture that all but a special subclass of groups have terraces remains open).