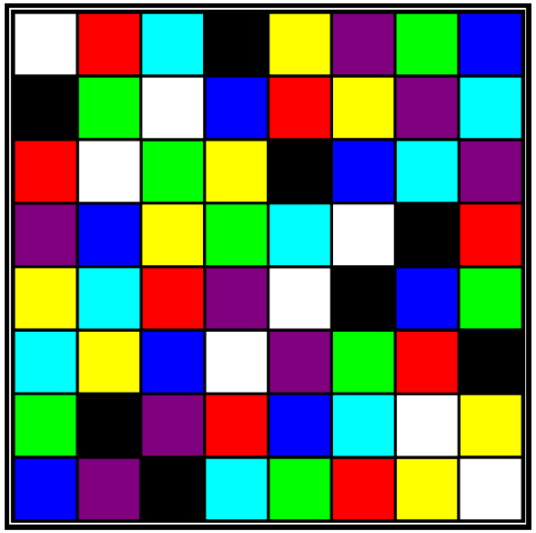
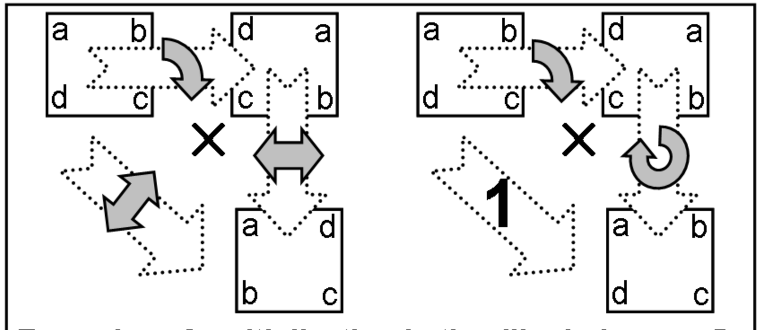
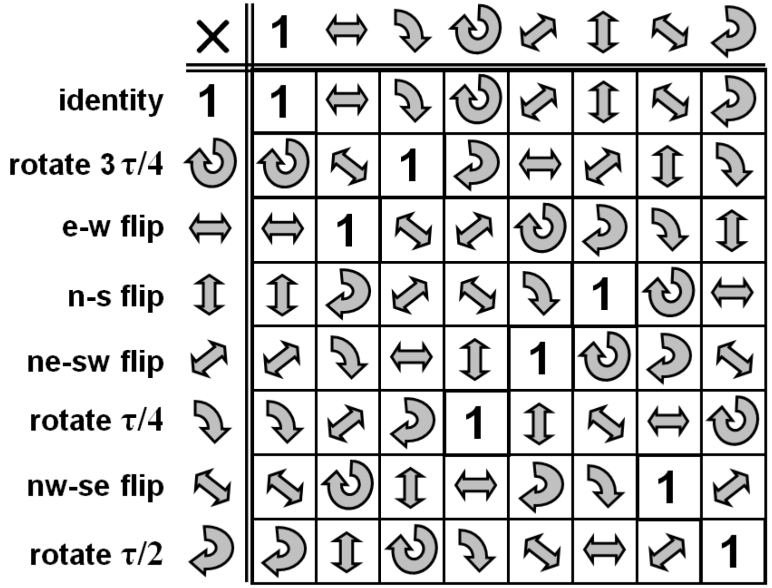
THEOREM OF THE DAY

Bailey's Theorem on Latin Squares *Let G be a finite group and let a and b be ordered listings of the elements of G. Then the Cayley table $L(a,b)$ is a quasi-complete Latin square if and only if a^{-1} and b are terraces for G.*





Examples of multiplication in the dihedral group D_8



A D_8 times-table

Colour-free version

Given a set of n treatments to test, an $n \times n$ *Latin square* is a grid in which each treatment appears exactly once in each row and column. In treatment testing, of a crop spray, for example, this factors out the influence of north-south and east-west geographical positioning (with respect to other fields, rivers etc). The Latin square constructed here is *quasi-complete* meaning that any two treatments appear adjacent to each other exactly twice, east to west, and exactly twice north to south. (*Complete* means every ordered pair occurs exactly once horizontally and once vertically, not true in the example above left.)

An ordered listing $a = (a_1, a_2, \ldots, a_n)$ of the elements of a group G is a *terrace* if every set $\{a_i, a_i^{-1}\}$ is represented twice in the sequence $a^* = (1, a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-1}^{-1}a_n)$. The inverse a^{-1} denotes the listing $(1, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$. For a square, let r denote its rotation through $\tau/4$ (a quarter circle) and f an east-west flip. The complete set of eight symmetries of the square constitutes the *dihedral group*, D_8, with terraces $a = (1, r, f, r^2 f, rf, r^3, r^3 f, r^2)$ and $b = (1, f, r, r^3, rf, r^2 f, r^3 f, r^2)$. The *Cayley table*, above right, multiples b (column headings) with a^{-1} (row headings). Both $\left(a^{-1}\right)^{-1} = a$ and b are terraces, so the theorem tells us that $L(a^{-1}, b)$ is a quasi-complete Latin square.

Not all Latin squares are group multiplication tables but R.A. Bailey, in a classic 1984 paper, argued that, for reasonable numbers of treatments, group tables were sufficiently prevalent to offer a practical source of experimental designs. She was then able to use the machinery of group theory to bring the issue of quasi-complete Latin square construction to a 'quasi-complete' resolution (her conjecture that all but a special subclass of groups have terraces remains open).

Web link: www.combinatorics.org/ojs/index.php/eljc/article/view/DS10

Further reading: *Latin Squares: New Developments in the Theory and Applications* by J. Dénes and A.D. Keedwell, North-Holland, 1991.