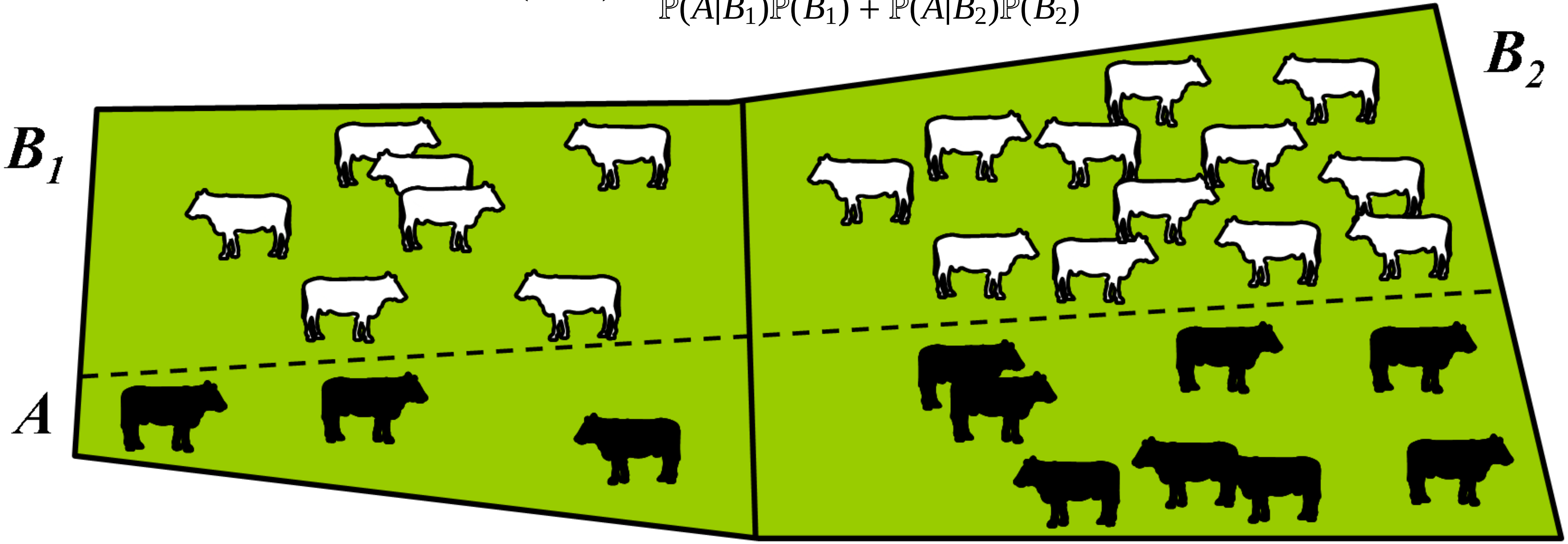
THEOREM OF THE DAY

Bayes' Theorem *Suppose a sample space S is partitioned into two non-empty parts B_1 and B_2. Then the conditional probability that a point in S satisfying some property A will also lie in B_1, is given by*

$$\mathbb{P}(B_1|\,A) = \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)}.$$



A farmer has herded all her thirty cows into two fields, imaginatively called B_1 and B_2. The black cows have conveniently migrated to the bottom of each field and form a subset A of the cows. Suppose we find a cow, what can we say about which field we are in? The probability of a cow being in field B_1, $\mathbb{P}(B_1)$, is $10/30 = 1/3$ and $\mathbb{P}(B_2) = 20/30 = 2/3$. Now suppose that the cow we found was a black cow, how does this affect the odds that we are in field B_1? The probability of the cow we found being black, *given* that we are in field B_1, is $\mathbb{P}(A|B_1) = 3/10$, since 3 of the 10 cows in B_1 belong in A; similarly $\mathbb{P}(A|B_2) = 8/20 = 2/5$. Bayes' theorem tells us that $\mathbb{P}(B_1|A) = \frac{3}{10}\cdot\frac{1}{3}\Big/\left(\frac{3}{10}\cdot\frac{1}{3}+\frac{2}{5}\cdot\frac{2}{3}\right) = \frac{3}{11}$; clearly correct, since 3 out of the 11 black cows are in field B_1. So the chance we are in B_1 drops from $\approx 33\%$ to $\approx 27\%$.

The theorem extends to a greater number of regions, $B_1, B_2, B_3, \ldots$, by extending the sum in the denominator in the obvious way; and to a single region, B_1, by replacing the denominator by $\mathbb{P}(A)$. The Rev. Thomas Bayes was one of the first to write about conditional probability. His work was published after his death in 1761 but lay for a long time forgotten. His theorem has eventually come to be used controversially to convert possibly subjective assessments of *prior* probability $\mathbb{P}(B_1)$ into a *posterior* probability $\mathbb{P}(B_1|A)$. This so-called *Bayesian* approach has sometimes been accused of applying the rigorous machinery of probability theory to inputs which may be guesswork or supposition.

Web link: www.dcs.qmul.ac.uk/~norman/papers/probability_puzzles/cancer.html.

Further reading: *Making Decisions, 2nd Ed.* by D.V. Lindley, John Wiley & Sons, 1985.