# THEOREM OF THE DAY

**Fisher's Inequality** *If a balanced incomplete block design is specified with parameters* $(v, b, r, k, \lambda)$ *then* $v \leq b$.

SUM =MMULT(C3:Q12;T3:AC17)

| | B0 | B1 | B2 | B3 | B4 | B5 | B6 | B7 | B8 | B9 | B10 | B11 | B12 | B13 | B14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| T1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| T2 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| T3 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| T4 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| T5 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| T6 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| T7 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| T8 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| T9 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |

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| | T0 | T1 | T2 | T3 | T4 | T5 | T6 | T7 | T8 | T9 |
|---|---|---|---|---|---|---|---|---|---|---|
| B0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| B1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| B2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| B3 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| B4 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| B5 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| B6 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| B7 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| B8 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| B9 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | |
| B10 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| B11 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| B12 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| B13 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| B14 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |

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| | T0 | T1 | T2 | T3 | T4 | T5 | T6 | T7 | T8 | T9 |
|---|---|---|---|---|---|---|---|---|---|---|
| T0 | 6 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| T1 | 2 | 6 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| T2 | 2 | 2 | 6 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| T3 | 2 | 2 | 2 | 6 | 2 | 2 | 2 | 2 | 2 | 2 |
| T4 | 2 | 2 | 2 | 2 | 6 | 2 | 2 | 2 | 2 | 2 |
| T5 | 2 | 2 | 2 | 2 | 2 | 6 | 2 | 2 | 2 | 2 |
| T6 | 2 | 2 | 2 | 2 | 2 | 2 | 6 | 2 | 2 | 2 |
| T7 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 6 | 2 | 2 |
| T8 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 6 | 2 |
| | | | | | 2 | 2 | 2 | 2 | 2 | 6 |

=MMULT(C3:Q12;T3:AC17)

| v | b | r | k | λ |
|---|---|---|---|---|
| 10 | 15 | 6 | 4 | 2 |

Row **Ti**, Column **Bj** = 1 if and only if treatment **Ti** is in block **Bj**

In a balanced incomplete block design, or **BIBD**, a set of $v$ **treatments** are selected, with repetition, to form $b$ **blocks**, each being a set of cardinality $k$, where $k < v$ (whence 'incomplete'), in such a way that

1. every treatment occurs in exactly $r$ blocks ('first order balance'; implies the equality $bk = rv$); and
2. every unordered pair of treatments occurs in exactly $\lambda$ blocks ('second order balance'; implies $\lambda(v-1) = r(k-1)$ and $r > \lambda$).

A BIBD may be represented by an incidence matrix $M$, as illustrated above left in the OpenOffice Calc screenshot; if this is multiplied by its transpose $M^T$ (centre) then the balance conditions are represented in the resulting $v \times v$ matrix, $MM^T$ (right), which has $r$ in each diagonal positions and $\lambda$ everywhere else (encircled we see how the $r = 6$ blocks containing treatment T1 match, exactly $\lambda = 2$ times, those containing treatment T5).

Add to row 1 of $MM^T$ each other row. Subtract column 1 in the resulting matrix from each other column. The result is the matrix shown on the right whose determinant is the product of its diagonal elements, which is $(r+(v-1)\lambda)(r-\lambda)^{v-1}$. Since $r > \lambda$ this is non-zero; in other words $\mathrm{rank}(MM^T) = v$. But $\mathrm{rank}(MM^T) \leq \mathrm{rank}(M) \leq \min(v, b)$. So we must have $\min(v, b) = v$, i.e. $v \leq b$, and this proves Fisher's Inequality. The inequality allows us, for example, to bound block size $k$, given $v$ and $\lambda$: condition 1 above gives $k \leq r$ whence condition 2 gives $\lambda(v-1) \geq k(k-1)$, and now solving for $v$ gives $k \leq \frac{1}{2}(1 + \sqrt{1 + 4\lambda(v-1)})$.

$$\begin{pmatrix} r+(v-1)\lambda & 0 & 0 & \ldots & 0 & 0 \\ \lambda & r-\lambda & 0 & \ldots & 0 & 0 \\ \lambda & 0 & r-\lambda & 0 & \ldots & 0 \\ \vdots & & \vdots & & & \vdots \\ \lambda & 0 & 0 & 0 & \ldots & r-\lambda \end{pmatrix}$$

Ronald Fisher's fundamental property of BIBDs dates from 1940. The above proof is due to Raj Chandra Bose (1949).

**Web link:** btravers.weebly.com/uploads/6/7/2/9/6729909/combinatorial_design_slides_student.pdf.

**Further reading:** *Combinatorial Designs and Tournaments* by Ian Anderson, OUP, 1997.