## THEOREM OF THE DAY

Fisher's Inequality If a balanced incomplete block design is specified with parameters ( $v, b, r, k, \lambda$ ) then $v \leq b$.


In a balanced incomplete block design, or BIBD, a set of $v$ treatments are selected, with repetition, to form $b$ blocks, each being a set of cardinality $k$, where $k<v$ (whence 'incomplete'), in such a way that

1. every treatment occurs in exactly $r$ blocks ('first order balance'; implies the equality $b k=r v$ ); and
2. every unordered pair of treatments occurs in exactly $\lambda$ blocks ('second order balance'; implies $\lambda(v-1)=r(k-1)$ and $r>\lambda)$.

A BIBD may be represented by an incidence matrix $M$, as illustrated above left in the OpenOffice Calc screenshot; if this is multiplied by its transpose $M^{T}$ (centre) then the balance conditions are represented in the resulting $v \times v$ matrix, $M M^{\mathrm{T}}$ (right), which has $r$ in each diagonal positions and $\lambda$ everywhere else (encircled we see how the $r=6$ blocks containing treatment T1 match, exactly $\lambda=2$ times, those containing treatment T5).
Add to row 1 of $M M^{\mathrm{T}}$ each other row. Subtract column 1 in the resulting matrix from each other column. The result is the matrix shown on the right whose determinant is the product of its diagonal elements, which is $(r+(v-1) \lambda)(r-\lambda)^{v-1}$. Since $r>\lambda$ this is non-zero; in other words $\operatorname{rank}\left(M M^{\mathrm{T}}\right)=v$. But $\operatorname{rank}\left(M M^{\mathrm{T}}\right) \leq \operatorname{rank}(M) \leq \min (v, b)$. So we must have $\min (v, b)=v$, i.e. $v \leq b$, and this proves Fisher's Inequality. The inequality allows us, for example, to bound block size $k$, given $v$ and $\lambda$ : condition 1 above gives $k \leq r$ whence condition 2 gives $\lambda(v-1) \geq k(k-1)$, and now
$\left(\begin{array}{cccccc}r+(v-1) \lambda & 0 & 0 & \ldots & 0 & 0 \\ \lambda & r-\lambda & 0 & \ldots & 0 & 0 \\ \lambda & 0 & r-\lambda & 0 & \ldots & 0 \\ \vdots & & \vdots & & & \vdots \\ \lambda & 0 & 0 & 0 & \ldots & r-\lambda\end{array}\right)$ solving for $v$ gives $k \leq \frac{1}{2}(1+\sqrt{1+4 \lambda(v-1)})$.

Ronald Fisher's fundamental property of BIBDs dates from 1940. The above proof is due to Raj Chandra Bose (1949).

## Web link: math.mit.edu/~1mlovasz/oddtowngood.pdf

Further reading: Combinatorial Designs and Tournaments by Ian Anderson, Clarendon Press, 1997 cre

