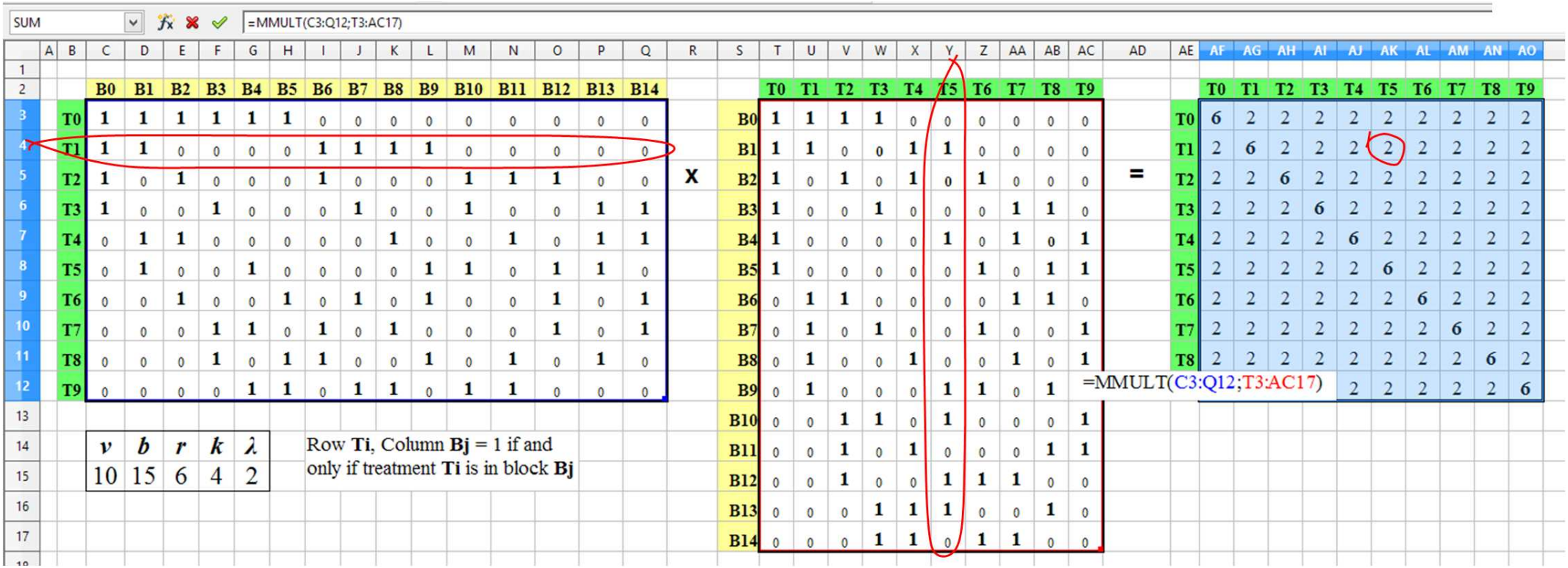
THEOREM OF THE DAY

Fisher's Inequality *If a balanced incomplete block design is specified with parameters* (v, b, r, k, λ) *then* $v \leq b$.



In a balanced incomplete block design, or **BIBD**, a set of v **treatments** are selected, with repetition, to form b **blocks**, each being a set of cardinality k, where $k < v$ (whence 'incomplete'), in such a way that

1. every treatment occurs in exactly r blocks ('first order balance'; implies the equality $bk = rv$); and
2. every unordered pair of treatments occurs in exactly λ blocks ('second order balance'; implies $\lambda(v-1) = r(k-1)$ and $r > \lambda$).

A BIBD may be represented by an incidence matrix M, as illustrated above left in the OpenOffice Calc screenshot; if this is multiplied by its transpose M^{T} (centre) then the balance conditions are represented in the resulting $v \times v$ matrix, MM^{T} (right), which has r in each diagonal positions and λ everywhere else (encircled we see how the $r = 6$ blocks containing treatment T1 match, exactly $\lambda = 2$ times, those containing treatment T5).

Add to row 1 of MM^{T} each other row. Subtract column 1 in the resulting matrix from each other column. The result is the matrix shown on the right whose determinant is the product of its diagonal elements, which is $(r+(v-1)\lambda)(r-\lambda)^{v-1}$. Since $r > \lambda$ this is non-zero; in other words $\mathrm{rank}(MM^{\mathrm{T}}) = v$. But $\mathrm{rank}(MM^{\mathrm{T}}) \leq \mathrm{rank}(M) \leq \min(v, b)$. So we must have $\min(v, b) = v$, i.e. $v \leq b$, and this proves Fisher's Inequality. The inequality allows us, for example, to bound block size k, given v and λ: condition 1 above gives $k \leq r$ whence condition 2 gives $\lambda(v-1) \geq k(k-1)$, and now solving for v gives $k \leq \frac{1}{2}(1 + \sqrt{1 + 4\lambda(v-1)})$.

$$\begin{pmatrix} r+(v-1)\lambda & 0 & 0 & \dots & 0 & 0 \\ \lambda & r-\lambda & 0 & \dots & 0 & 0 \\ \lambda & 0 & r-\lambda & 0 & \dots & 0 \\ \vdots & & \vdots & & & \vdots \\ \lambda & 0 & 0 & 0 & \dots & r-\lambda \end{pmatrix}$$

Ronald Fisher's fundamental property of BIBDs dates from 1940. The above proof is due to Raj Chandra Bose (1949).

Web link: opentext.uleth.ca/Combinatorics/sect_designs-FisherIneq.html
Further reading: *Combinatorial Designs and Tournaments* by Ian Anderson, Clarendon Press, 1997.