## **THEOREM OF THE DAY**

**Fisher's Inequality** If a balanced incomplete block design is specified with parameters  $(v, b, r, k, \lambda)$ then  $v \leq b$ .

SUM			~	fx	*	V	=M	MULT	(C3:Q	12;T3	AC17	)																														
	AB	С	D		E	F	G	Н	I	J	K		L	М	N	0	Р	Q	R	S	T	U	V	W	X	Y,	Z	AA	AB	AC	AD	AE	AF	AG	AH	Al	AJ	AK	AL	AM	AN	AO
1 2	_	BO	B	1 1	B2	<b>B3</b>	<b>B4</b>	B5	<b>B6</b>	<b>B</b> 7	B	8 I	B9	<b>B10</b>	B11	B12	B13	B14			TO	<b>T1</b>	T2	<b>T3</b>	T4	15	T6	<b>T</b> 7	<b>T8</b>	Т9			TO	Tl	T2	T3	T4	T5	T6	<b>T</b> 7	<b>T</b> 8	<b>T9</b>
3	TO	1	1		1	1	1	1	0	0			0	0	0	0	0	0		B		1	1	1	0	0	0	0	0	0		TO	6	2	2	2	2	2	2	2	2	2
47~	11	1	1		0	0	0	0	1	1	1		1	0	0	0	0	0	>	B	1	1	0	0	1	1	0	0	0	0		Tl	2	6	2	2	2	2	2	2	2	2
5	Т2	1	0		1	0	0	0	1	0	0		0	1	1	1	0	0	X	B	1	0	1	0	1	0	1	0	0	0	=	Т2	2	2	6	2	2	2	2	2	2	2
6	Т3	1	0		0	1	0	0	0	1	0		0	1	0	0	1	1		B	1	0	0	1	0	0	0	1	1	0		<b>T</b> 3	2	2	2	6	2	2	2	2	2	2
7	Т4	0	1	n j	1	0	0	0	0	0	1		0	0	1	0	1	1		B	1	0	0	0	0	1	0	1	0	1		T4	2	2	2	2	6	2	2	2	2	2
8	Т5	0	1		0	0	1	0	0	0	0		1	1	0	1	1	0		B	1	0	0	0	0	0	1	0	1	1		Т5	2	2	2	2	2	6	2	2	2	2
9	<b>T6</b>	0	0		1	0	0	1	0	1	0		1	0	0	1	0	1		B	0	1	1	0	0	0	0	1	1	0		T6	2	2	2	2	2	2	6	2	2	2
10	<b>T</b> 7	0	0		0	1	1	0	1	0	1		0	0	0	1	0	1		B	0	1	0	1	0	0	1	0	0	1		<b>T</b> 7	2	2	2	2	2	2	2	6	2	2
11	<b>T</b> 8	0	0	1	0	1	0	1	1	0	0		1	0	1	0	1	0		B	0	1	0	0	1	0	0	1	0	1		<b>T8</b>		2	2	2	2	2	2	2	6	2
12	Т9	0	0		0	0	1	1	0	1	1		0	1	1	0	0	0		B	0	1	0	0	0	1	1	0	1	=N	MULT	(C3	:Q1	2;T3	AC	17)	2	2	2	2	2	6
13																				<b>B1</b>	0	0	1	1	0	1	0	0	0	1												
14		v	b		r	k	λ		Row Ti, Column $\mathbf{Bj} = 1$ if and											B1	0	0	1	0	1	0	0	0	1	1												
15		10	) 15	5	6	4	2		only if treatment Ti is in block Bj										B1	0	0	1	0	0	1	1	1	0	0													
16																				<b>B1</b>	0	0	0	1	1	1	0	0	1	0												
17																				B1.	4 0	0	0	1	1	0	1	1	0	0												

In a balanced incomplete block design, or **BIBD**, a set of v treatments are selected, with repetition, to form b blocks, each being a set of cardinality k, where k < v(whence 'incomplete'), in such a way that

1. every treatment occurs in exactly r blocks ('first order balance'; implies the equality bk = rv); and

2. every unordered pair of treatments occurs in exactly  $\lambda$  blocks ('second order balance'; implies  $\lambda(v-1) = r(k-1)$  and  $r > \lambda$ ).

A BIBD may be represented by an incidence matrix M, as illustrated above left in the OpenOffice Calc screenshot; if this is multiplied by its transpose  $M^{T}$  (centre) then the balance conditions are represented in the resulting  $v \times v$  matrix,  $MM^{T}$  (right), which has r in each diagonal positions and  $\lambda$  everywhere else (encircled we see how the r = 6 blocks containing treatment T1 match, exactly  $\lambda = 2$  times, those containing treatment T5).

Add to row 1 of  $MM^{T}$  each other row. Subtract column 1 in the resulting matrix from each other column. The result is the matrix shown on the right whose determinant is the product of its diagonal elements, which is  $(r+(v-1)\lambda)(r-\lambda)^{v-1}$ . Since  $r > \lambda$  this is non-zero; in other words rank $(MM^{T}) = v$ . But rank $(MM^{T}) \le rank(M) \le min(v, b)$ . So we must have  $\min(v, b) = v$ , i.e.  $v \le b$ , and this proves Fisher's Inequality. The inequality allows us, for example, to bound block size k, given v and  $\lambda$ : condition 1 above gives  $k \leq r$  whence condition 2 gives  $\lambda(v-1) \geq k(k-1)$ , and now solving for v gives  $k \leq \frac{1}{2}(1 + \sqrt{1 + 4\lambda(v - 1)})$ .

Ronald Fisher's fundamental property of BIBDs dates from 1940. The above proof is due to Raj Chandra Bose (1949).



Web link: math.mit.edu/~lmlovasz/oddtowngood.pdf Further reading: Combinatorial Designs and Tournaments by Ian Anderson, Clarendon Press, 1997. Created by Robin Whitty for www.theoremoftheday.org

 $r + (v - 1)\lambda$ 

 $r - \lambda = 0$ 

... 0

 $0 \quad r-\lambda \quad 0 \quad \dots$ 

0

0