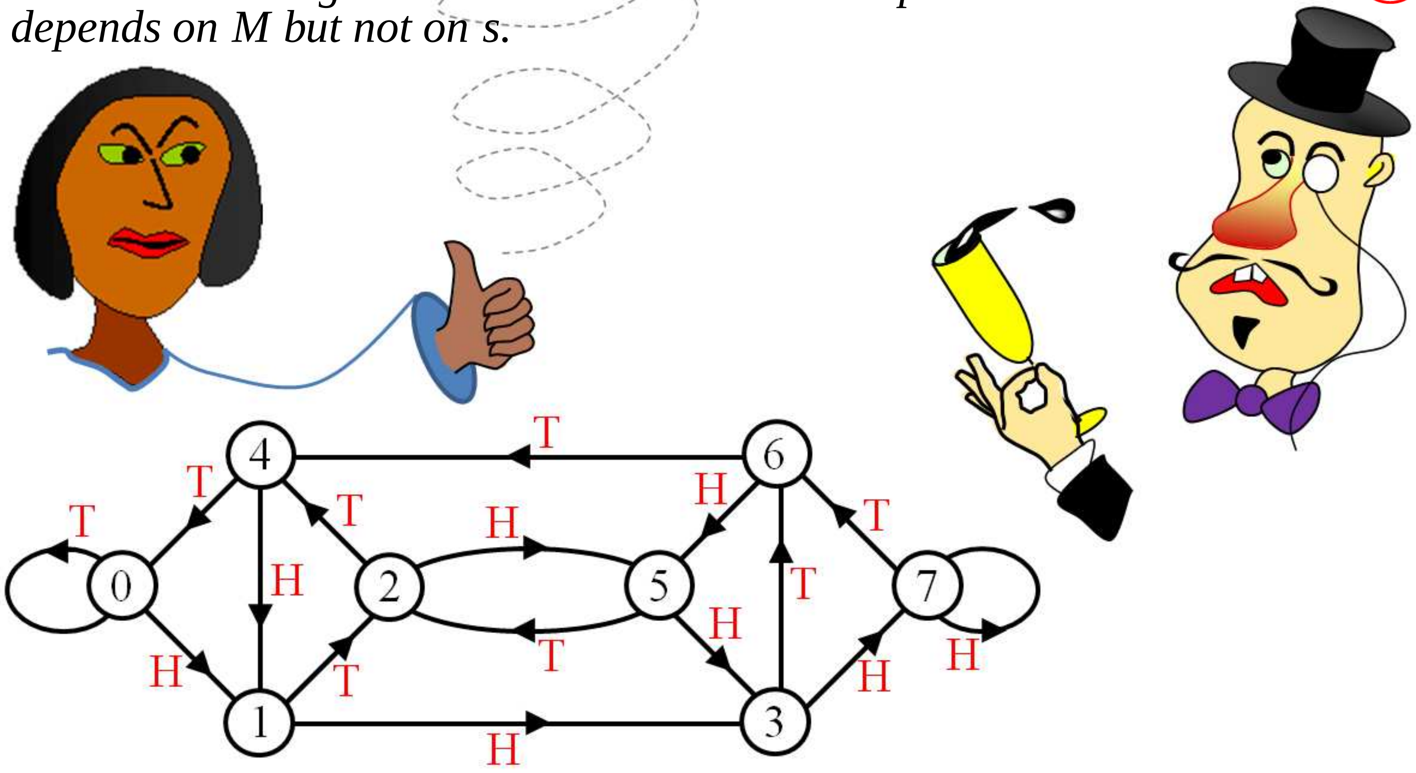
THEOREM OF THE DAY

Kemeny's Constant *Let s be a state in a finite-state Markov chain M which has a unique stationary distribution. If state t is chosen at random according to this distribution then the expected time to reach t from state s is a constant K which depends on M but not on s.*

Alice's Casino
In Alice's Casino a gambler, call him Bob, purchases for \$7 a number from the set $\{0,\ldots,7\}$ which he then uses in the following game:

1. Alice also selects from $\{0,\ldots,7\}$, choosing an element at random (see below).
2. If Alice has the same number as Bob the game is over.
3. When Alice's number differs from Bob's, she pays Bob \$1 and tosses a coin. Bob doubles his number and, if the toss is heads, adds 1. Finally he reduces his number modulo 8.
4. Steps 2 and 3 repeat until the game ends.



For example, perhaps Bob chooses number 2. Alice perhaps chooses number 6. Play might now proceed as follows:

$$2 \xrightarrow{\text{T}} 4 \xrightarrow{\text{H}} 1 \xrightarrow{\text{H}} 3 \xrightarrow{\text{H}} 7 \xrightarrow{\text{H}} 7 \xrightarrow{\text{T}} 6.$$

Bob has doubled six times and so he is down by \$1. He consoles himself: "Maybe next game I will double fifty times!" but asks himself "Maybe 2 was an unlucky choice of starting number for me?" Advice for Bob comes from Markov chain theory. The casino game may be represented by the above directed graph, in which Bob is trying to get from his chosen number to Alice's number by following edges according to Alice's coin tosses. Provided Alice's coin is fair we may replace each H and T in the graph with a 1/2, the probability that Bob follows either exit edge from each vertex.

The result is a Markov chain and this chain indeed has a stationary distribution: it is $\left(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$, meaning that each vertex ('state' in the chain) has equal chance of being reached by a random walk around the chain. And now we can see that, provided Alice chooses uniformly at random from $\{0,\ldots,7\}$, she is choosing a target state t at random according to the stationary distribution. Then Kemeny's constant K means that there are no lucky numbers for Bob: over many games any starting state will, on average, take as many doublings to reach Alice's random targets. How many doublings? For this chain the answer is exactly $K = 7$, which means the casino only breaks even! Suppose Alice rigs the game by tossing, *whenever Bob is in state 2*, a different coin which shows heads with probability only 1/3? The result is a different Markov chain, with a different stationary distribution: $\left(\frac{7}{50},\frac{7}{50},\frac{3}{25},\frac{3}{25},\frac{7}{50},\frac{1}{10},\frac{3}{25},\frac{3}{25}\right)$. Provided Alice uses this new distribution to choose her target number, Kemeny's constant will still be a constant, but for this different chain its value has decreased by 1/25 to $K = 174/25$. Now Alice has a very slight advantage over Bob, and over many games she will pay him back about \$6.96 for every \$7 he bets.

John G. Kemeny introduced his constant in his classic textbook with J. Laurie Snell, *Finite Markov Chains*, 1960.

Web link: math.dartmouth.edu/~doyle/docs/kc/kc.pdf.
Further reading: *Introduction to Probability, 2nd Ed.* by C.M. Grinstead & J.L. Snell, AMS, 1997.