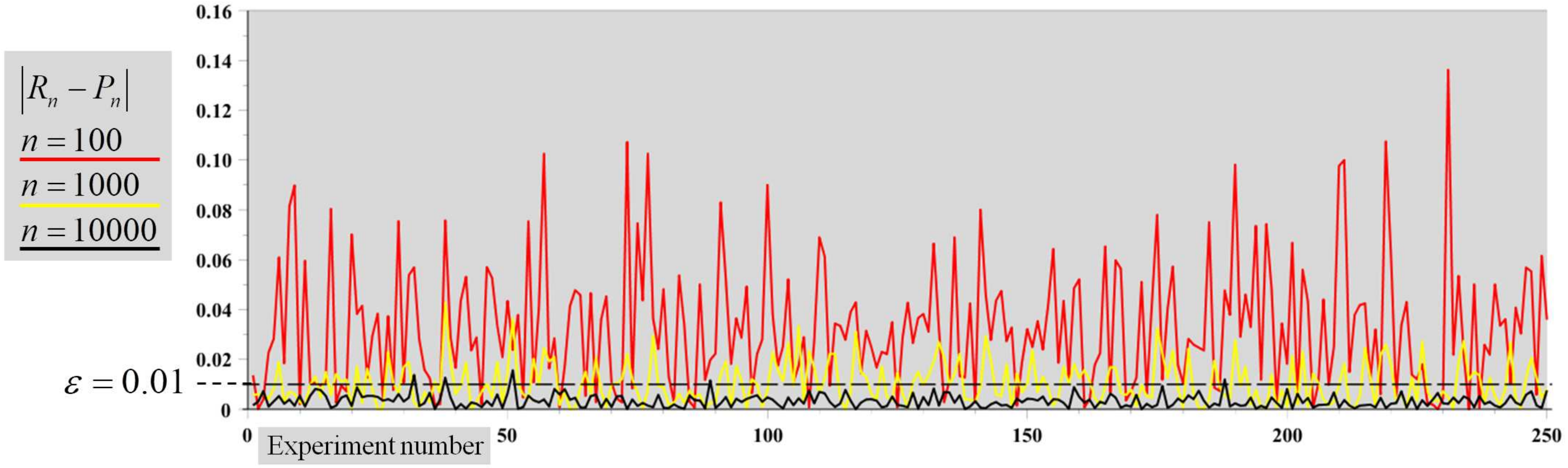
THEOREM OF THE DAY

The Law of Large Numbers *Consider an event A which, in a sequence of independent trials, occurs with probabilities of occurrence $p_i, i = 1, 2, \ldots$. Let P_n denote the mean of the first n probabilities, $\frac{1}{n}\sum_{i=1}^{n} p_i$, and let R_n denote the relative frequency of occurrence of A in the first n trials, i.e. $\frac{1}{n}\times$(number of times that A occurs). Then, for any $\varepsilon > 0$, as $n \to \infty$, $\mathbb{P}(|R_n - P_n| < \varepsilon) \to 1$.*



An experiment was defined as follows: 10^4 pairs of integers in the range $0, \ldots, 999$ were generated pseudorandomly. Each pair defined (1) an event A, the event that first entry $\geq$ second entry, and (2) a probability: (first entry + 1)/1000. This was taken as the probability of event A occurring. For example, for a first entry of 169, event A occurs for 170 out of the 1000 possible second entries.
Now the absolute value of $R_n - P_n$ was calculated as specified in the theorem, for $n = 100, 1000$ and 10000: i.e., the first 100 of our 10^4 pairs, the first 1000 and the whole sequence of 10^4. The experiment was repeated 250 times and the values of $|R_n - P_n|$ were plotted as shown above. A value of $\varepsilon = 0.01$ was chosen, plotted as a dotted line. This was found to be exceeded by $|R_n - P_n|$ in 80% of the experiments for $n = 100$, in about 44% of the experiments for $n = 1000$ and in just 2% of the experiments for $n = 10000$.

The term ‘Law of Large Numbers’ applies to a large and general body of results which say, more or less precisely and in more or less generality, that many random occurrences happening together can collectively produce a non-random effect. This particular theorem is due to Siméon-Denis Poisson in 1837 who, in referring to it, was the first to use the phrase ‘law of large numbers.’ It may be thought of as justifying the ‘frequentist’ view that relative frequency is, in the long run, a good proxy for probability.

Web link: eom.springer.de/l/l057720.htm
Further reading: *Mathematical Statistics with Applications, 6th ed.* by D. Wackerly, W. Mendenhall III and R. Scheaffer, Duxbury, 2002.