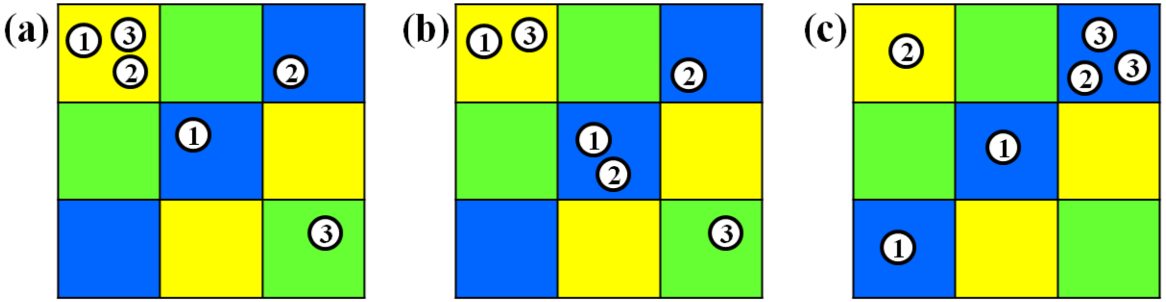
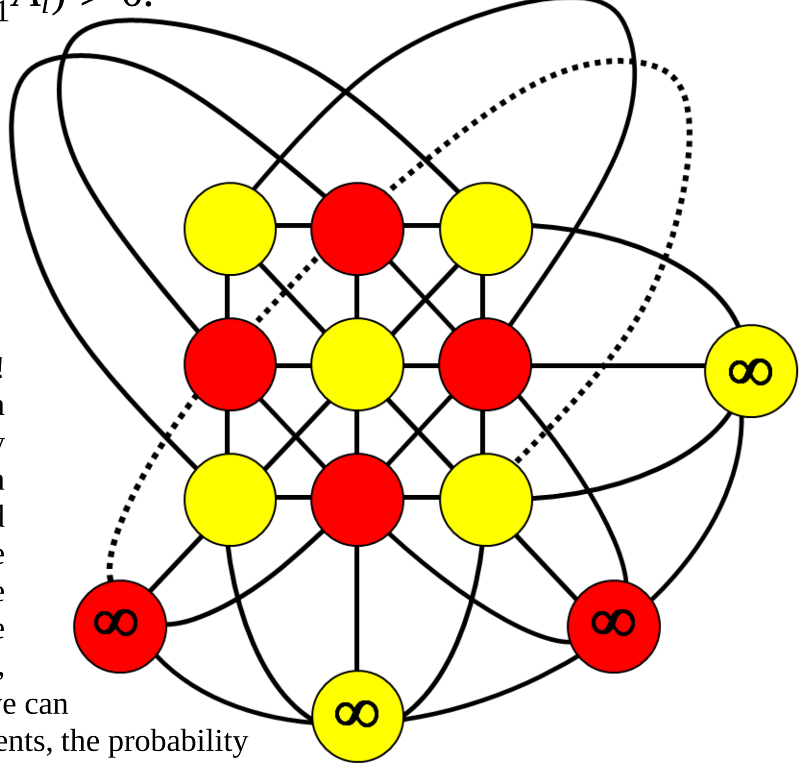
THEOREM OF THE DAY

The Lovász Local Lemma *Let $A_1, \ldots A_t$ be events in a probability space, with every event being independent of all except at most d others. Suppose, for some non-negative real number p satisfying $p \le 1/e(d+1)$ ($e = 2.718\ldots$), we have $\mathbb{P}(A_i) < p$ for all i, $1 \le i \le t$. Then $\mathbb{P}(\cap_{i=1}^{t} \overline{A}_i) > 0$.*





A cautionary tale: if you mis-read the hypotheses of a lemma you may mis-apply its conclusion! Suppose n players each privately commit to two (possibly equal) cells on a 3×3 grid. Then they reveal their choices by placing tokens on the grid, as illustrated above. Their stake money is shared evenly between all nonempty cells which are not chosen by two or more players. In game (a) above, $n = 3$ and all three players get an equal share, each having one non-shared location; in game (b), the stake is shared equally between players 2 and 3; in game (c), the stake is shared 2/3 to player 1 and 1/3 to player 2. The key events in this game are A_{ij}: the two players i and j share a cell. What is the probability $\mathbb{P}(A_{ij})$ of the event A_{ij}? We can calculate this as $77/225 \approx 0.34$, and this is independent of any non-identical event A_{kl}: the choices made by, say, players 1 and 2 have no influence on player 3's choices, so $\mathbb{P}(A_{12} \cap A_{13}) = \mathbb{P}(A_{12}) \times \mathbb{P}(A_{13})$, etc. So we can take $d = 0$ in the Local Lemma, can we not? Then since $\mathbb{P}(A_{ij}) \approx 0.34 < 1/e(0+1) \approx 0.37$ for all events, the probability that no event occurs, that is, $\cap_{1 \le i < j \le n} \overline{A}_{ij}$, is greater than zero. And this means that there is at least one way in which all n players may choose different cells from each other...? **Absolutely not true!** If $n \ge 10$ this conclusion is self-evidently impossible. Our mistake is also self-evident! Event A_{jk} is more likely to happen if A_{ij} and A_{ik} occur. In game (a), above, for example, A_{12} occurs because players 1 and 2 both choose the top left-hand cell; and since A_{13} occurs for the same reason, A_{23} occurs automatically. It is possible that A_{12} and A_{13} occur without A_{23}, as in game (b) for example; nevertheless, we can calculate that $\mathbb{P}(A_{12} \cap A_{13} \cap A_{23}) \approx 0.08$, almost twice $\mathbb{P}(A_{12}) \times \mathbb{P}(A_{13}) \times \mathbb{P}(A_{23})$: the events are only *pairwise* independent, and this is not enough for the Local Lemma.

A correct (in fact the original) application is illustrated above right. An r-uniform, k-regular hypergraph is a collection of r-element subsets (*edges*) of a set whose every element (*vertex*) itself lies in exactly k edges. The edges in our illustration are the thirteen 4-point lines emanating from the vertices marked ∞ (the dotted line is an example). In 1992 Carsten Thomassen proved: *the vertices of a k-uniform, k-regular hypergraph, $k \ge 4$, may be 2-coloured so that no edge is monochromatic.* This is a difficult result but $k = 9$ is an easy consequence of the Local Lemma: take the probability space to be the random 2-colourings of the vertices and the events A_i to be monochromatic edges. Then $\mathbb{P}(A_i) = 1/2^{k-1}$ and $d \le k(k-1)$, and for $k \ge 9$ we have $1/2^{k-1} < 1/e(k(k-1)+1)$.

The Local Lemma was devised by Paul Erdős and László Lovász in 1973 to tackle problems in hypergraph colouring. They used the bound $p < 1/4d$; the stronger (for $d > 2$) bound given here was derived by Joel Spencer from an 'asymmetric' version of the Lemma which he attributed to Lovász.

Web link: faculty.cs.tamu.edu/klappi/csce658-s18/: scroll down to Lecture Notes (4th-last)

Further reading: *The Probabilistic Method, 3rd edition* by Noga Alon and Joel H. Spencer, WileyBlackwell, 2008.