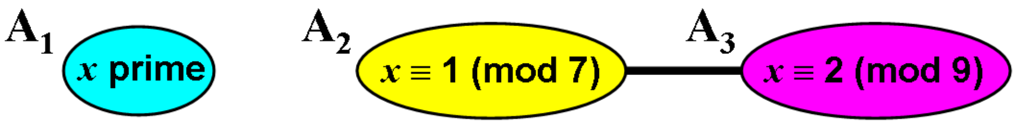
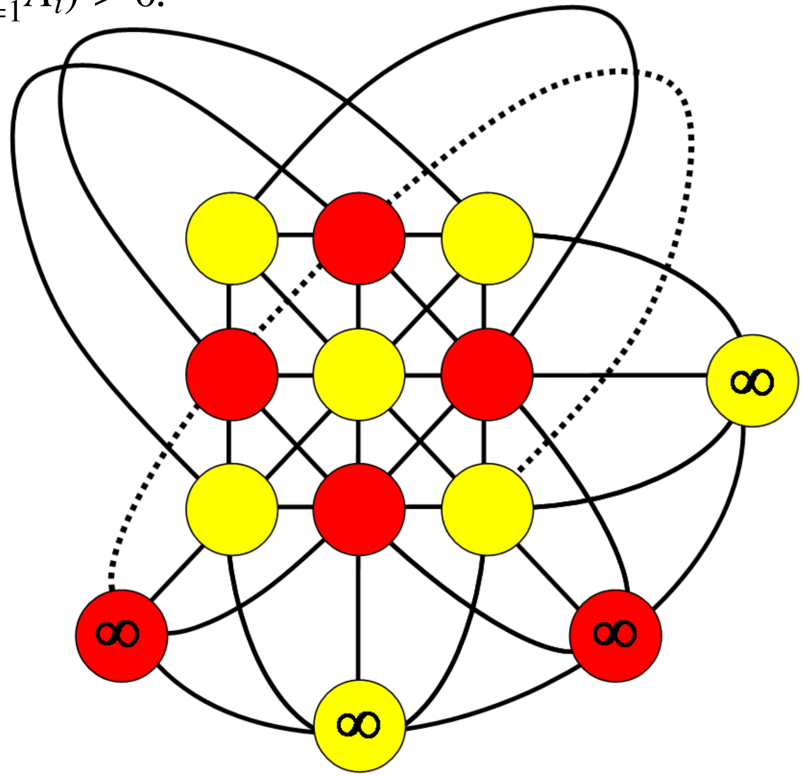
THEOREM OF THE DAY

The Lovász Local Lemma *Let $A_1, \ldots A_t$ be events in a probability space, with every event being independent of all but at most d others. Suppose, for some non-negative real number p satisfying $p \leq 1/e(d+1)$, $(e = 2.718\ldots)$ we have $\mathbb{P}(A_i) < p$ for all i, $1 \leq i \leq t$. Then $\mathbb{P}(\cap_{i=1}^{t}\overline{A}_i) > 0$.*

Pairwise dependency graph for three events (for sets X and Y)



		prime	≡ 1 (mod 7)	≡ 2 (mod 9)
X	{478, ..., 525}	479, 487, 491, 499, 503, 509, 521, 523	484, 491, 498, 505, 512, 519	479, 488, 497, 506, 515, 524
Y	{387,..., 434}	389, 397, 401, 409, 419, 421, 431, 433	393, 400, 407, 414, 421, 428	389, 398, 407, 416, 425, 434



We start, above, by proving a property of a set of 48 consecutive integers which is completely trivial but demonstrates how the Local Lemma is applied — and how it should not be! The probability space consists of choosing x uniformly at random from the set. For set X, we have $\mathbb{P}(A_1) = \mathbb{P}(x \text{ prime}) = 1/6$; $\mathbb{P}(A_2) = \mathbb{P}(x \equiv 1 \pmod 7) = \mathbb{P}(A_3) = \mathbb{P}(x \equiv 2 \pmod 9) = 1/8$. As indicated in the dependency graph, A_1 is independent of A_2 and A_3, i.e. $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = (1/6).(1/8)$ *and* $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1).\mathbb{P}(A_2 \cap A_3) = 0$. Now $d = 1$ and $1/6 < 1/(2e)$, so the Local Lemma guarantees that X contains a composite number congruent to neither $1 \pmod 7$ nor $2 \pmod 9$. Set Y also contains such a number, but for Y, A_1 is only *pairwise* independent from A_2 and A_3; $\mathbb{P}(A_1 \cap A_2 \cap A_3) = 0$ but $\mathbb{P}(A_1).\mathbb{P}(A_2 \cap A_3) = (1/6).(1/48)$. So the Local Lemma does not apply for set Y.

A non-trivial (in fact the original) application is illustrated above right. A 4-uniform, 4-regular hypergraph is a collection of 4-element hypergraph *edges*, being subsets of a set whose each element itself lies in 4 edges. The subsets here are the thirteen 4-point lines emanating from the points marked ∞ (the dotted line is an example). With the probability space being the random 2-colourings of the points and the events being monochromatic edges, a simple application of the Local Lemma shows that any 9-uniform, 9-regular hypergraph has a 2-colouring with no monochromatic edges. (The value of 9 was reduced to 4, as illustrated here using the projective plane of order 3, by Carsten Thomassen, in 1992, by other methods).

The Local Lemma was devised by Paul Erdős and László Lovász in 1975 to tackle the problem of hypergraph 2-colourings. It has since, with various subtle modifications, found applications in many parts of mathematics, computer science and physics.

Web link: www.essex.ac.uk/maths/pdfs/square2/SQUARE2_issue15.pdf (a nice undergraduate magazine — see p. 2 ff.)

Further reading: *The Probabilistic Method, 3rd edition* by Noga Alon and Joel H. Spencer, WileyBlackwell, 2008.