THEOREM OF THE DAY

Distribution of local maxima in random samples *Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation of $\{1, \ldots, n\}$ and let k be a positive integer. The k-local maxima of π are defined to be the maximum values taken by length k subsequences of π: i.e., the set $\{\max(\pi_i, \ldots, \pi_{i+k-1}), 1 \le i \le n-k+1\}$. Denote by $f_k(n,m)$ the number of permutations π having exactly m distinct k-local maxima. Let $v_k(x,y)$ be the generating function for the f_k defined as $v_k(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_k(i,j)x^i y^j/i!$. Then the discrete distribution of the probabilities $\Pr(\pi$ has m local maxima$)$, $m = 0, \ldots, n$, is given by*

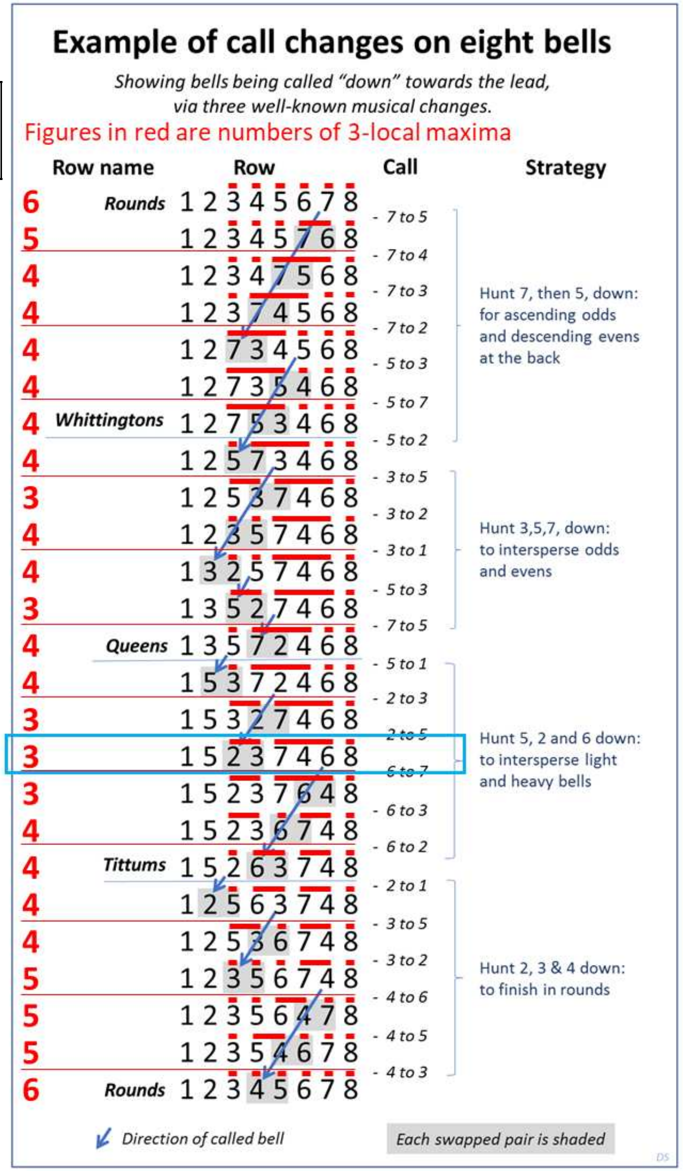
$$\frac{1}{n!}\frac{\partial^n v}{\partial x^n}\bigg|_{x=0}.$$

Moreover, $v_k(x,y)$ satisfies the partial differential equation

$$\frac{\partial v}{\partial x} = yv^2 + (1-y)\left(1 + 2x + \ldots + (k-1)x^{k-2}\right), \qquad (1)$$

with boundary conditions $v_k(0,y) = \frac{\partial v}{\partial x}\big|_{x=0} = 1$.

Adapted from an image by Dougsim at en.wikipedia.org/wiki/Change_ringing



This theorem is about a statistic for random samples: move a length k 'window' along a sample of size n. How often does the maximum value in the window change? The question is adequately answered in terms of k-local maxima in permutations, illustrated on the right using a collection of permutations found in change ringing of church bells. For the boxed permutation two-thirds down, for example, there are three updates, indicated by the red bars, corresponding to window 1 ([152]), window 3 ([237]) and window 6 ([468]). How likely (for a random permutation of $\{1, \ldots, 8\}$) would 3 updates be? The theorem derives the answer from the coefficient of y^3 when x is set to zero in the 8-th partial derivative $\partial^8 v/\partial x^8$. The differential equation (1), for $k = 3$, is $\partial v/\partial x = yv^2 + (1-y)(1+2x)$ and this gives an ingenious method for finding the higher derivatives. Differentiate both sides: $\partial^2 v/\partial x^2 = 2yv\partial v/\partial x + 2(1-y)$. Setting $x = 0$ and using the boundary conditions, $\partial^2 v/\partial x^2|_{x=0} = 2y \times 1 \times 1 + 2(1-y) = 2$. Differentiate again: $\partial^3 v/\partial x^3 = 2y(\partial v/\partial x)^2 + 2yv\partial^2 v/\partial x^2$. At $x = 0$, this evaluates to $\partial^3 v/\partial x^3|_{x=0} = 2y \times 1^2 + 2y \times 1 \times 2 = 6y$. If we continue this process we eventually find that $\partial^8 v/\partial x^8|_{x=0} = 2016y^2 + 18624y^3 + 17376y^4 + 2112y^5 + 192y^6$. The coefficients sum to 8! and the probability that a random permutation of $\{1, \ldots, 8\}$ has three 3-local maxima is $18624/8! \approx 0.46$. In our illustration 3-local maxima certainly do not constitute nearly half of the permutations, but this is not surprising since the permutations in change ringing are generated in a very systematic manner!

A simple formula for the mean number of k-local maxima can be derived by differentiating once again but with respect to y: this multiplies each term by the number of local maxima it is counting. So then setting $y = 1$ gives the usual formula for expected value. And happily the partial derivative, which is $\frac{1}{n!}\frac{\partial}{\partial y}\left(\frac{\partial^n v}{\partial x^n}\right)\Big|_{x=0,y=1}$, can be shown to simplify to $(2n-k+1)/(k+1)$.

This theorem was published in 1957 by T.L. Austin, R.E. Fagen, T.A. Lehrer and W. F. Penney.

Web link: www.informit.com/articles/article.aspx?p=2243840.

Further reading: Concrete Mathematics by R.L. Graham, D.E. Knuth and O. Patashnik, Addison Wesley, 1994.