



THEOREM OF THE DAY

Distribution of local maxima in random samples Let $\pi = (\pi_1, \dots, \pi_n)$ be a permutation of $\{1, \dots, n\}$ and let k be a positive integer. The k -local maxima of π are defined to be the maximum values taken by length k subsequences of π : i.e., the set $\{\max(\pi_i, \dots, \pi_{i+k-1}), 1 \leq i \leq n - k + 1\}$. Denote by $f_k(n, m)$ the number of permutations π having exactly m distinct k -local maxima. Let $v_k(x, y)$ be the generating function for the f_k defined as $v_k(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f_k(i, j) x^i y^j / i!$. Then the discrete distribution of the probabilities $\Pr(\pi \text{ has } m \text{ local maxima}), m = 0, \dots, n$, is given by

$$\frac{1}{n!} \frac{\partial^n v}{\partial x^n} \Big|_{x=0}$$

Adapted from an image by Dougsim at en.wikipedia.org/wiki/Change_ringing

Moreover, $v_k(x, y)$ satisfies the partial differential equation

$$\frac{\partial v}{\partial x} = yv^2 + (1 - y)(1 + 2x + \dots + (k - 1)x^{k-2}), \quad (1)$$

with boundary conditions $v_k(0, y) = \frac{\partial v}{\partial x} \Big|_{x=0} = 1$.

This theorem is about a statistic for random samples: move a length k 'window' along a sample of size n . How often does the maximum value in the window change? The question is adequately answered in terms of k -local maxima in permutations, illustrated on the right using a collection of permutations found in change ringing of church bells. For the boxed permutation two-thirds down, for example, there are three updates, indicated by the red bars, corresponding to window 1 ([152]), window 3 ([237]) and window 6 ([468]). How likely (for a random permutation of $\{1, \dots, 8\}$) would 3 updates be? The theorem derives the answer from the coefficient of y^3 when x is set to zero in the 8-th partial derivative $\partial^8 v / \partial x^8$. The differential equation (1), for $k = 3$, is $\partial v / \partial x = yv^2 + (1 - y)(1 + 2x)$ and this gives an ingenious method for finding the higher derivatives. Differentiate both sides: $\partial^2 v / \partial x^2 = 2yv \partial v / \partial x + 2(1 - y)$. Setting $x = 0$ and using the boundary conditions, $\partial^2 v / \partial x^2 \Big|_{x=0} = 2y \times 1 \times 1 + 2(1 - y) = 2$. Differentiate again: $\partial^3 v / \partial x^3 = 2y(\partial v / \partial x)^2 + 2yv \partial^2 v / \partial x^2$. At $x = 0$, this evaluates to $\partial^3 v / \partial x^3 \Big|_{x=0} = 2y \times 1^2 + 2y \times 1 \times 2 = 6y$. If we continue this process we eventually find that $\partial^8 v / \partial x^8 \Big|_{x=0} = 2016y^2 + 18624y^3 + 17376y^4 + 2112y^5 + 192y^6$. The coefficients sum to 8! and the probability that a random permutation of $\{1, \dots, 8\}$ has three 3-local maxima is $18624 / 8! \approx 0.46$. In our illustration 3-local maxima certainly do not constitute nearly half of the permutations, but this is not surprising since the permutations in change ringing are generated in a very systematic manner!

A simple formula for the mean number of k -local maxima can be derived by differentiating once again but with respect to y : this multiplies each term by the number of local maxima it is counting. So then setting $y = 1$ gives the usual formula for expected value. And happily the partial derivative, which is $\frac{1}{n!} \frac{\partial}{\partial y} \left(\frac{\partial^n v}{\partial x^n} \right) \Big|_{x=0, y=1}$, can be shown to simplify to $(2n - k + 1) / (k + 1)$.

This theorem was published in 1957 by T.L. Austin, R.E. Fagen, T.A. Lehr and W. F. Penney.

Web link: www.informit.com/articles/article.aspx?p=2243840.

Further reading: *Concrete Mathematics* by R.L. Graham, D.E. Knuth and O. Patashnik, Addison Wesley, 1994.

Example of call changes on eight bells

Showing bells being called "down" towards the lead, via three well-known musical changes.

Figures in red are numbers of 3-local maxima

Row name	Row	Call	Strategy
6	Rounds 1 2 3 4 5 6 7 8	- 7 to 5	Hunt 7, then 5, down: for ascending odds and descending evens at the back
5	1 2 3 4 5 7 6 8	- 7 to 4	
4	1 2 3 4 7 5 6 8	- 7 to 3	
4	1 2 3 7 4 5 6 8	- 7 to 2	
4	1 2 7 3 4 5 6 8	- 5 to 3	
4	1 2 7 3 5 4 6 8	- 5 to 7	
4	Whittingtons 1 2 7 5 3 4 6 8	- 5 to 2	Hunt 3,5,7, down: to intersperse odds and evens
4	1 2 5 7 3 4 6 8	- 3 to 5	
3	1 2 5 3 7 4 6 8	- 3 to 2	
4	1 2 3 5 7 4 6 8	- 3 to 1	
4	1 3 2 5 7 4 6 8	- 5 to 3	
3	1 3 5 2 7 4 6 8	- 7 to 5	
4	Queens 1 3 5 7 2 4 6 8	- 5 to 1	Hunt 5, 2 and 6 down: to intersperse light and heavy bells
4	1 5 3 7 2 4 6 8	- 2 to 3	
3	1 5 3 2 7 4 6 8	2 to 5	
3	1 5 2 3 7 4 6 8	5 to 7	
3	1 5 2 3 7 6 4 8	- 6 to 3	
4	1 5 2 3 6 7 4 8	- 6 to 2	
4	Tittums 1 5 2 6 3 7 4 8	- 2 to 1	Hunt 2, 3 & 4 down: to finish in rounds
4	1 2 5 6 3 7 4 8	- 3 to 5	
4	1 2 5 3 6 7 4 8	- 3 to 2	
5	1 2 3 5 6 7 4 8	- 4 to 6	
5	1 2 3 5 6 4 7 8	- 4 to 5	
5	1 2 3 5 4 6 7 8	- 4 to 3	
6	Rounds 1 2 3 4 5 6 7 8		

Direction of called bell (indicated by blue arrows pointing down)

Each swapped pair is shaded

