



# THEOREM OF THE DAY

**Distribution of local maxima in random samples** Let  $\pi = (\pi_1, \dots, \pi_n)$  be a permutation of  $\{1, \dots, n\}$  and let  $k$  be a positive integer. The  $k$ -local maxima of  $\pi$  are defined to be the maximum values taken by length  $k$  subsequences of  $\pi$ : i.e., the set  $\{\max(\pi_i, \dots, \pi_{i+k-1}), 1 \leq i \leq n - k + 1\}$ . Denote by  $f_k(n, m)$  the number of permutations  $\pi$  having exactly  $m$  distinct  $k$ -local maxima. Let  $v_k(x, y)$  be the generating function for the  $f_k$  defined as  $v_k(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f_k(i, j) x^i y^j / i!$ . Then the discrete distribution of the probabilities  $\Pr(\pi \text{ has } m \text{ local maxima}), m = 0, \dots, n$ , is given by

$$\frac{1}{n!} \frac{\partial^n v}{\partial x^n} \Big|_{x=0}$$

Adapted from an image by Dougsim at [en.wikipedia.org/wiki/Change\\_ringing](https://en.wikipedia.org/wiki/Change_ringing)

Moreover,  $v_k(x, y)$  satisfies the partial differential equation

$$\frac{\partial v}{\partial x} = yv^2 + (1 - y)(1 + 2x + \dots + (k - 1)x^{k-2}), \quad (1)$$

with boundary conditions  $v_k(0, y) = \frac{\partial v}{\partial x} \Big|_{x=0} = 1$ .

This theorem is about a statistic for random samples: move a length  $k$  'window' along a sample of size  $n$ . How often does the maximum value in the window change? The question is adequately answered in terms of  $k$ -local maxima in permutations, illustrated on the right using a collection of permutations found in change ringing of church bells. For the boxed permutation two-thirds down, for example, there are three updates, indicated by the red bars, corresponding to window 1 ([152]), window 3 ([237]) and window 6 ([468]). How likely (for a random permutation of  $\{1, \dots, 8\}$ ) would 3 updates be? The theorem derives the answer from the coefficient of  $y^3$  when  $x$  is set to zero in the 8-th partial derivative  $\partial^8 v / \partial x^8$ . The differential equation (1), for  $k = 3$ , is  $\partial v / \partial x = yv^2 + (1 - y)(1 + 2x)$  and this gives an ingenious method for finding the higher derivatives. Differentiate both sides:  $\partial^2 v / \partial x^2 = 2yv \partial v / \partial x + 2(1 - y)$ . Setting  $x = 0$  and using the boundary conditions,  $\partial^2 v / \partial x^2 \Big|_{x=0} = 2y \times 1 \times 1 + 2(1 - y) = 2$ . Differentiate again:  $\partial^3 v / \partial x^3 = 2y(\partial v / \partial x)^2 + 2yv \partial^2 v / \partial x^2$ . At  $x = 0$ , this evaluates to  $\partial^3 v / \partial x^3 \Big|_{x=0} = 2y \times 1^2 + 2y \times 1 \times 2 = 6y$ . If we continue this process we eventually find that  $\partial^8 v / \partial x^8 \Big|_{x=0} = 2016y^2 + 18624y^3 + 17376y^4 + 2112y^5 + 192y^6$ . The coefficients sum to  $8!$  and the probability that a random permutation of  $\{1, \dots, 8\}$  has three 3-local maxima is  $18624 / 8! \approx 0.46$ . In our illustration 3-local maxima certainly do not constitute nearly half of the permutations, but this is not surprising since the permutations in change ringing are generated in a very systematic manner!

A simple formula for the mean number of  $k$ -local maxima can be derived by differentiating once again but with respect to  $y$ : this multiplies each term by the number of local maxima it is counting. So then setting  $y = 1$  gives the usual formula for expected value. And happily the partial derivative, which is  $\frac{1}{n!} \frac{\partial}{\partial y} \left( \frac{\partial^n v}{\partial x^n} \right) \Big|_{x=0, y=1}$ , can be shown to simplify to  $(2n - k + 1) / (k + 1)$ .

This theorem was published in 1957 by T.L. Austin, R.E. Fagen, T.A. Lehr and W. F. Penney.

**Web link:** [www.informit.com/articles/article.aspx?p=2243840](http://www.informit.com/articles/article.aspx?p=2243840).

**Further reading:** *Concrete Mathematics* by R.L. Graham, D.E. Knuth and O. Patashnik, Addison Wesley, 1994.

## Example of call changes on eight bells

Showing bells being called "down" towards the lead, via three well-known musical changes.

Figures in red are numbers of 3-local maxima

Row name	Row	Call	Strategy
6	Rounds 1 2 3 4 5 6 7 8	- 7 to 5	
5	1 2 3 4 5 7 6 8	- 7 to 4	
4	1 2 3 4 7 5 6 8	- 7 to 3	
4	1 2 3 7 4 5 6 8	- 7 to 2	
4	1 2 7 3 4 5 6 8	- 5 to 3	
4	1 2 7 3 5 4 6 8	- 5 to 7	
4	Whittingtons 1 2 7 5 3 4 6 8	- 5 to 2	
4	1 2 5 7 3 4 6 8	- 3 to 5	
3	1 2 5 3 7 4 6 8	- 3 to 2	
4	1 2 3 5 7 4 6 8	- 3 to 1	Hunt 3,5,7, down: to intersperse odds and evens
4	1 3 2 5 7 4 6 8	- 5 to 3	
3	1 3 5 2 7 4 6 8	- 7 to 5	
4	Queens 1 3 5 7 2 4 6 8	- 5 to 1	
4	1 5 3 7 2 4 6 8	- 2 to 3	
3	1 5 3 2 7 4 6 8	2 to 5	
3	1 5 2 3 7 4 6 8	5 to 7	Hunt 5, 2 and 6 down: to intersperse light and heavy bells
3	1 5 2 3 7 6 4 8	- 6 to 3	
4	1 5 2 3 6 7 4 8	- 6 to 2	
4	Tittums 1 5 2 6 3 7 4 8	- 2 to 1	
4	1 2 5 6 3 7 4 8	- 3 to 5	
4	1 2 5 3 6 7 4 8	- 3 to 2	
5	1 2 3 5 6 7 4 8	- 4 to 6	
5	1 2 3 5 6 4 7 8	- 4 to 5	
5	1 2 3 5 4 6 7 8	- 4 to 3	Hunt 2, 3 & 4 down: to finish in rounds
6	Rounds 1 2 3 4 5 6 7 8		

Direction of called bell (indicated by blue arrows in the original image)

Each swapped pair is shaded

