THEOREM OF THE DAY

The Total Probability Theorem Suppose a sample space $S$ is partitioned into $n$ non-empty parts $B_1, B_2, \ldots, B_n$, $n \geq 1$. Then, for any event $A$,

$$P(A) = \sum P(A \cap B_i) = \sum P(A|B_i)P(B_i).$$

The deuce rule in tennis provides a well-known illustration of how the Total Probability Theorem is used in so-called ‘next-step analysis’. Federer is playing Nadal, the possible events following a deuce score of 40-40 being listed far right, with their effect on play being depicted near right. Suppose that Federer wins each point independently with probability $p$, and that Nadal wins with probability $q = 1 - p$. What is the probability $P(F)$ that Federer wins a game from deuce? The events $F_1$ and $N_1$ are clearly mutually exclusive and exhaustive — that is they partition $S = \{\text{outcomes of next point}\}$. So

$$P(F) = P(F|F_1)P(F_1) + P(F|N_1)P(N_1) = P(F|F_1)p + P(F|N_1)q.$$  

Resolving the unknowns in equation (1) illustrates the extension of the theorem to conditional probabilities. Denoting $P(F)$ by $v$:

$$P(F|F_1) = P(F|F_1 \cap F_2)P(F_2|F_1) + P(F|F_1 \cap N_2)P(N_2|F_1) = 1.p + v.q,$$

since the joint event $F_1 \cap F_2$ wins the game for Federer with probability 1 while $F_1 \cap N_2$ takes us back to deuce. For $P(F|N_1)$ we get

$$P(F|N_1) = v.p + 0.q.$$  

Substituting (2) and (3) back into equation (1) gives $v = (p + vq)p + vpq$. Rearranging and using $1 = p + q = (p + q)^2 = q^2 + 2pq + q^2$, we get Federer’s chance of winning: $v = p^2/(p^2 + q^2)$, while Nadal’s is $1 - v = q^2/(p^2 + q^2)$. Plotting these curves (above right) reveals that the deuce rule has the effect of exaggerating slightly the winning chances of a stronger player.

The notions of conditional probability and of summing mutually exclusive probabilities appear in the writings of Bayes (1764).


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