THEOREM OF THE DAY

The Total Probability Theorem *Suppose a sample space S is partitioned into n non-empty parts $B_1, B_2, \ldots, B_n$, $n \geq 1$. Then, for any event A,*

$$\mathbb{P}(A) = \sum \mathbb{P}(A \cap B_i) = \sum \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

The *deuce rule* in tennis provides a well-known illustration of how the Total Probability Theorem is used in so-called 'next-step analysis'. Federer is playing Nadal, the possible events following a deuce score of 40-40 being listed far right, with their effect on play being depicted near right. Suppose that Federer wins each point independently with probability p, and that Nadal wins with probability $q = 1 - p$. What is the probability $\mathbb{P}(F)$ that Federer wins a game from deuce? The events F_1 and N_1 are clearly mutually exclusive and exhaustive — that is they partition S = {outcomes of next point}. So

$$\begin{aligned}\mathbb{P}(F) &= \mathbb{P}(F|F_1)\mathbb{P}(F_1) + \mathbb{P}(F|N_1)\mathbb{P}(N_1)\\ &= \mathbb{P}(F|F_1)p + \mathbb{P}(F|N_1)q. \qquad (1)\end{aligned}$$

Resolving the unknowns in equation (1) illustrates the extension of the theorem to conditional probabilities. Denoting $\mathbb{P}(F)$ by v:

$$\begin{aligned}\mathbb{P}(F|F_1) &= \mathbb{P}(F|F_1 \cap F_2)\mathbb{P}(F_2|F_1)\\ &\quad + \mathbb{P}(F|F_1 \cap N_2)\mathbb{P}(N_2|F_1)\\ &= 1.p + v.q, \qquad (2)\end{aligned}$$

since the joint event $F_1 \cap F_2$ wins the game for Federer with probability 1 while $F_1 \cap N_2$ takes us back to deuce. For $\mathbb{P}(F|N_1)$ we get

$$\mathbb{P}(F|N_1) = v.p + 0.q. \qquad (3)$$

Substituting (2) and (3) back into equation (1) gives $v = (p + vq)p + vpq$. Rearranging and using $1 = p + q = (p + q)^2 = q^2 + 2pq + q^2$, we get Federer's chance of winning: $v = p^2/(p^2 + q^2)$, while Nadal's is $1 - v = q^2/(p^2 + q^2)$. Plotting these curves (above right) reveals that the deuce rule has the effect of exaggerating slightly the winning chances of a stronger player.

EVENTS

F = Federer wins from deuce

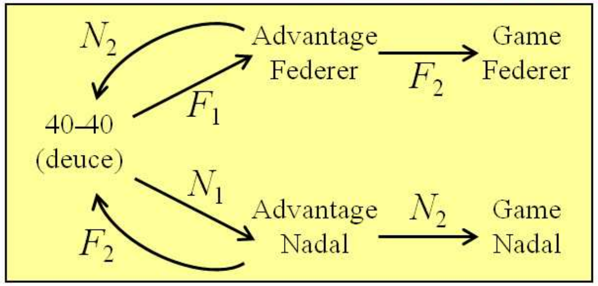
N = Nadal wins from deuce

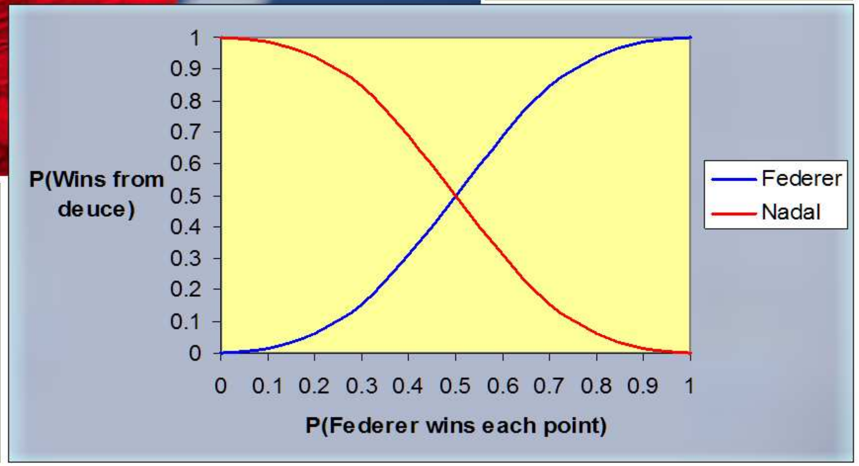
F_1 = Federer wins next point

N_1 = Nadal wins next point

F_2 = Federer wins point after next

N_2 = Nadal wins point after next





The notions of conditional probability and of summing mutually exclusive probabilities appear in the writings of Bayes (1764).

Web link: www.stat.auckland.ac.nz/~fewster/325/notes.php, Chapter 2. Roger Federer image reproduced from the *New York Daily News* with the kind permission of twitter.com/nydailynewspix.

Further reading: *Mathematical Statistics with Applications, 6th ed.* by D. Wackerly, W. Mendenhall III and R. Scheaffer, Duxbury, 2002.