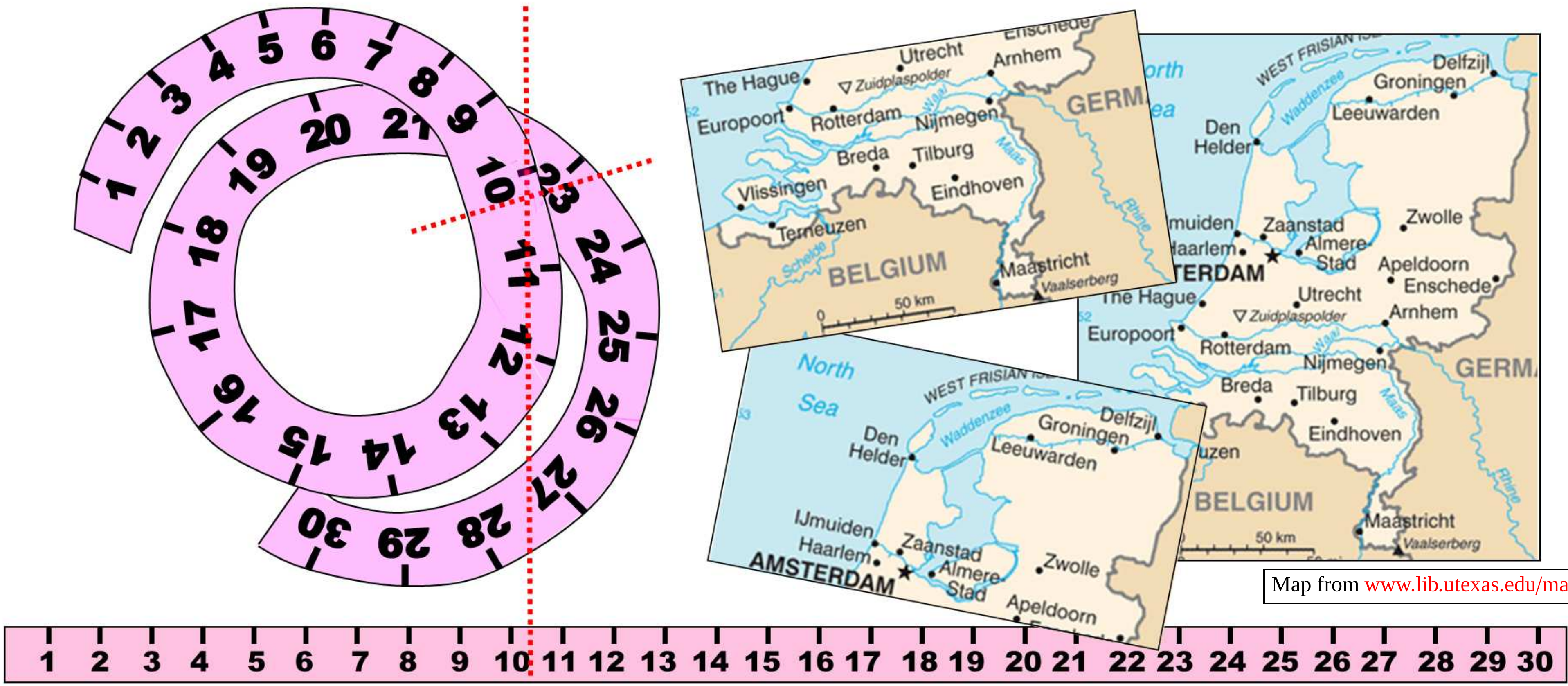
THEOREM OF THE DAY

Brouwer's Fixed Point Theorem *Let B denote a closed ball in n-dimensional Euclidean space and let $f : B \longrightarrow B$ be a continuous function. Then f has a fixed point: for some $x \in B$, $f(x) = x$.*



Map from www.lib.utexas.edu/maps/

In one dimension B can be taken to be the interval $[0, 1]$ consisting of all real numbers from 0 to 1. But the theorem applies to any set homeomorphic to B (so, closed, bounded, without holes). Brouwer's theorem can be visualised in terms of two tape measures (say, cloth ones, as used by dressmakers). The function f twists and folds one of the tape measures; which nevertheless retains at least one point aligned with the other measure. In two dimensions, take B to be the solid disk of radius one; this time the illustration consists of two copies of a map with one twisted and folded by f. Some location will again remain common to both maps. But f must be **continuous**: no breaks or jumps. If we cut a map of the Netherlands in half and invert top and bottom halves then no locations are left fixed.

This theorem derived from the application of topological methods to differential equations and was first proved, in three dimensions, in 1904 by the Latvian mathematician Piers Bohl. It was rediscovered and proved in the general case by L.E.J. Brouwer and, independently, Jacques Hadamard in 1910. Brouwer was the founding figure in the early 20th century movement to make mathematics *constructive*: denying, for example, that falsehood following from non-existence should constitute a proof of existence. There are many important non-constructive proofs of existence and this fixed point theorem is, ironically, one of them: it tells you there is a fixed point but offers no method, acceptable to a constructivist, by which it may be located.

Web link: rjlipton.wordpress.com/2009/04/ (the first entry). Brouwer and constructivism : www.theoremoftheday.org/Docs/Keller-Brouwer.pdf

Further reading: *Five Golden Rules: Great Theories of 20th Century Mathematics and Why They Matter* by John L Casti, Wiley, 1997.

Created by Robin Whitty for www.theoremoftheday.org