THEOREM OF THE DAY

Haken's Unknot Theorem *It is decidable whether a given knot, represented by a knot diagram, is equivalent to the unknot.*

A stylised version is shown here of a knot devised by Morwen Thistlethwaite. Unless you are a knot theorist or have strong visuospatial skills you will probably find it hard to see how it may be untangled, in three dimensions, to give a single loop—the so-called *unknot*. It is not even obvious that there is any procedure for deciding if this is possible in a finite number of steps: which is roughly what it means to say it is *decidable* to recognise the unknot. The famous Reidemeister moves will achieve the untangling if it is possible but they may greatly increase the number of crossings before reducing this number to zero; how can we tell if we are making progress or not?

The ideal would be to find a *complete invariant* for unknottedness: for example, we can try to paint each unbroken segment in a knot diagram with one of three colours so that at any crossing, all of the three colours, or just one, appear. If we can do this *and* use all three colours then the knot is called *three-colourable*. This property is independent of which diagram represents the knot and is called an invariant. It does not hold for the unknot, so a three-colouring proves non-equivalence to the unknot...

...but unluckily, there are plenty of knots other than the unknot which are not three-colourable.

So three-colourability fails to be a complete invariant for unknottedness.

In the 1920s, Hellmuth Kneser invented the theory of normal surfaces to help in the classification of 3-manifolds.

Wolfgang Haken showed in 1961 that Kneser's theory could be applied to decide whether or not two triangulated 3-manifolds containing incompressible surfaces were equivalent. This provides an algorithm for deciding if a knot is the unknot by applying Haken's theory to the complement of the knot. Haken's paper was 130 pages long and his algorithm has never been implemented. Since then other algorithms have been devised, notably by Joan Birman and Michael Hirsch, and at least partially implemented.

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Click the arrow to see the unknotting solution which Morwen Thistlethwaite has kindly provided (jpeg, 330KB).

Web link: gilkalai.wordpress.com/2012/04/10/

Further reading: *The Knot Book* by Colin Adams, AMS, 2004.