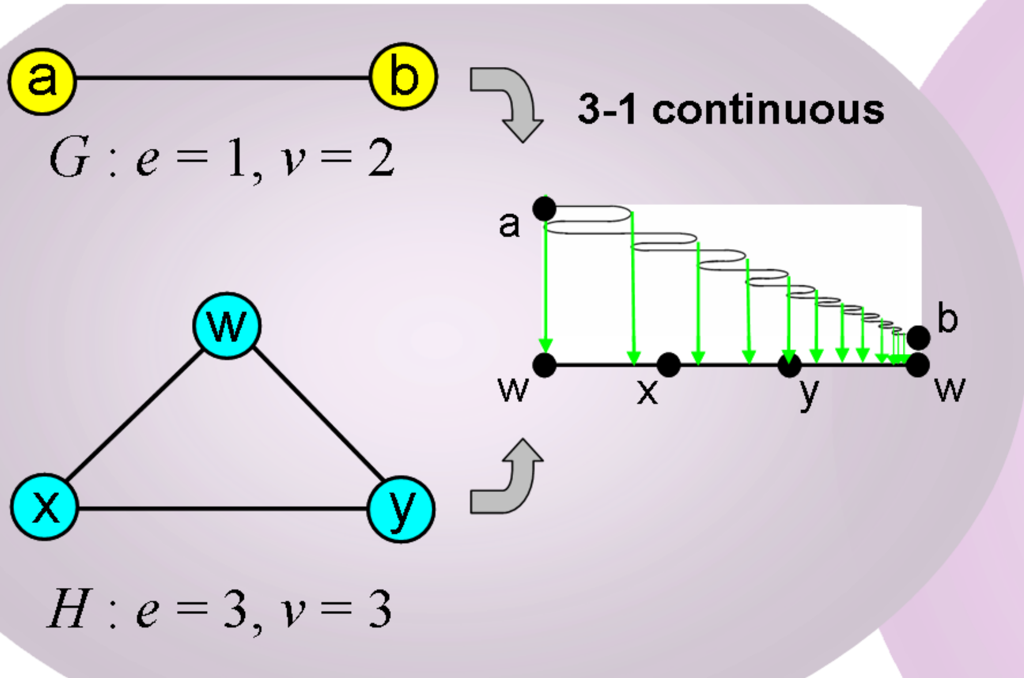
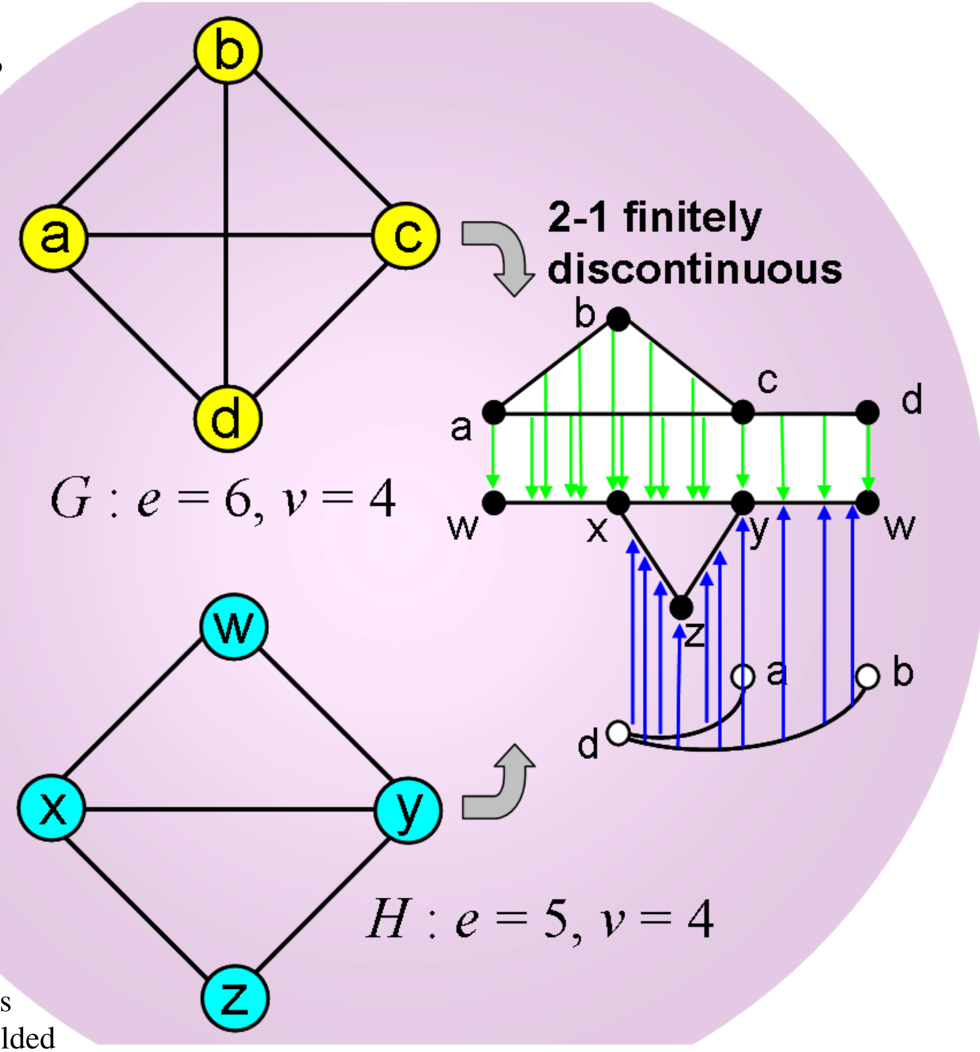
THEOREM OF THE DAY

Heath's Finitely Discontinuous Function Theorem *Let G and H be graphs whose edge and vertex sets have cardinality e_G and e_H, and v_G and v_H, respectively. There is a k-to-1 finitely discontinuous function from G onto H if and only if*

$$e_G - v_G \leq k(e_H - v_H),$$

with equality being required to hold in the case $k = 2$.





For the topologist, graph edges with distinct endpoints are homeomorphic images of the interval [0,1] and loops are homeomorphic images of the circle. For the graphs G and H above left we get a strict inequality $e_G - v_G = -1 < 0 = 2(e_H - v_H)$ so the theorem says it is impossible for H (i.e. the space of all points in all edges of H) to be the image of a finitely discontinuous function with every point the image of exactly $k = 2$ points of G. But for $k = 3$ a continuous function is possible: the line $wxyw$ represents H by identifying the points w; the line ab is repeatedly folded back on itself, the lengths of the folds tending to zero (a device affectionately termed a 'wiggle' by the mathematician Anthony Hilton). For the graphs G and H above right, the equality condition of the theorem is met so that $k = 2$ is possible. This is achieved in the drawing using three discontinuities (represented by unfilled vertices ○).

This surprising graph-theoretic answer to a subtle topological question, reminiscent of Kuratowski's Theorem, is due to Jo Heath, 1988. In the case where G is a single edge it generalises 1930s results of Orville G. Harrold and George Earl Schweigert.

Web link: arxiv.org/abs/math/9810129

Further reading: *The Foundations of Analysis: a Straightforward Introduction. Book 2 Topological Ideas* by K.G. Binmore, CUP, 1981.