Heath’s Finitely Discontinuous Function Theorem Let $G$ and $H$ be graphs whose edge and vertex sets have cardinality $e_G$ and $e_H$, and $v_G$ and $v_H$, respectively. There is a $k$-to-$1$ finitely discontinuous function from $G$ onto $H$ if and only if

$$e_G - v_G \leq k(e_H - v_H),$$

with equality being required to hold in the case $k = 2$.

For the topologist, graph edges with distinct endpoints are homeomorphic images of the interval $[0,1]$ and loops are homeomorphic images of the circle. For the graphs $G$ and $H$ above left we get a strict inequality $e_G - v_G = -1 < 0 = 2(e_H - v_H)$ so the theorem says it is impossible for $H$ (i.e. the space of all points in all edges of $H$) to be the image of a finitely discontinuous function with every point the image of exactly $k = 2$ points of $G$. But for $k = 3$ a continuous function is possible: the line $wxyw$ represents $H$ by identifying the points $w$; the line $ab$ is repeatedly folded back on itself, the lengths of the folds tending to zero (a device affectionately termed a ‘wiggle’ by the mathematician Anthony Hilton). For the graphs $G$ and $H$ above right, the equality condition of the theorem is met so that $k = 2$ is possible. This is achieved in the drawing using three discontinuities (represented by unfilled vertices $\circ$).

This surprising graph-theoretic answer to a subtle topological question, reminiscent of Kuratowski’s Theorem, is due to Jo Heath, 1988. In the case where $G$ is a single edge it generalises 1930s results of Orville G. Harrold and George Earl Schweigert.

Web link: arxiv.org/abs/math/9810129