THEOREM OF THE DAY
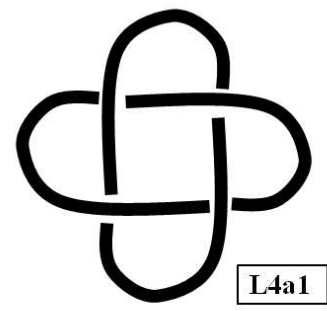
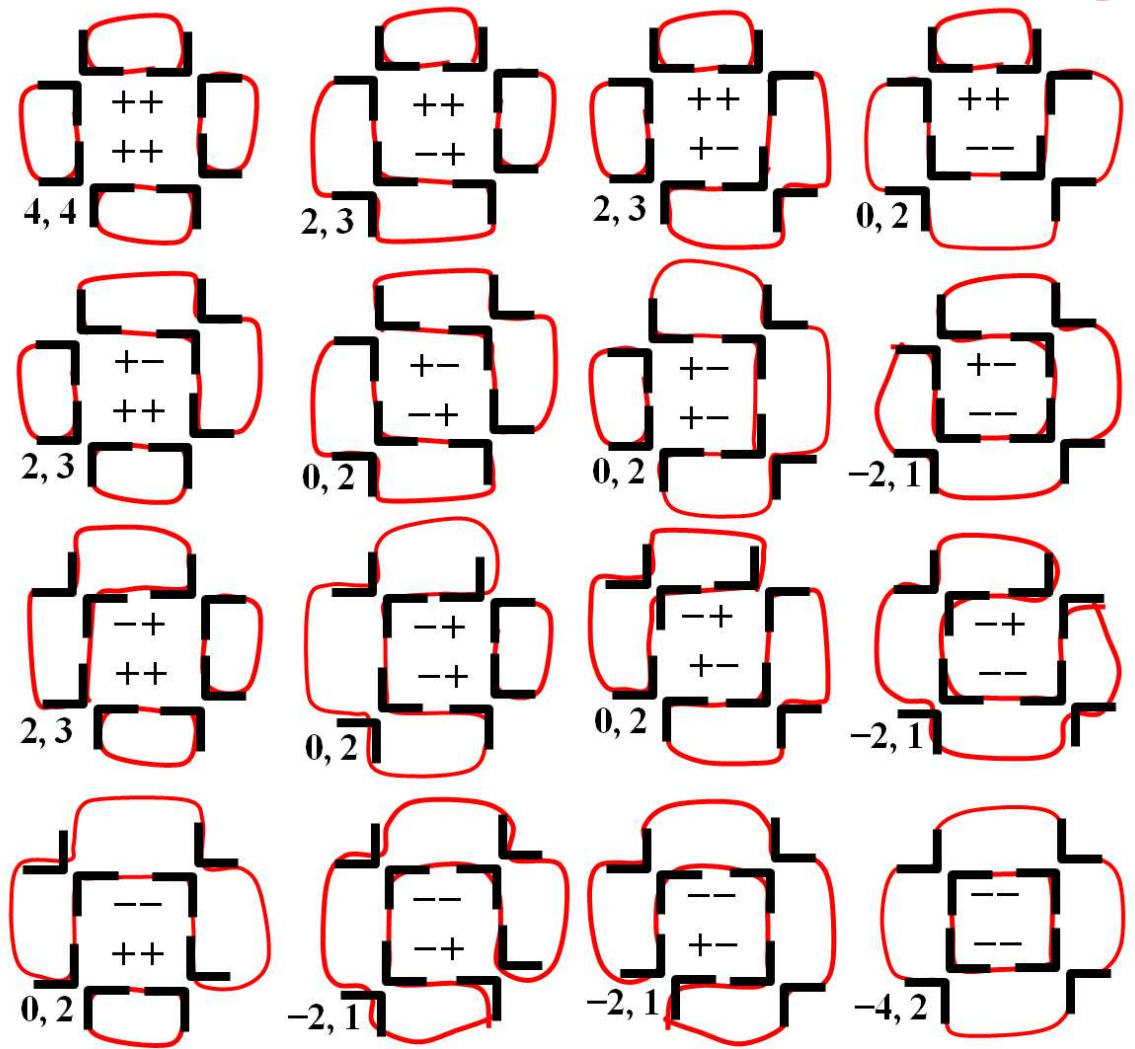**The Jones Knot Polynomial Theorem** *For a knot or link K, with writhe w(K) and Kauffman bracket ⟨K⟩, the Jones polynomial $J(K) = (-x)^{3w(K)}\langle K\rangle$ is a knot invariant.*

Solomon's Seal
Image: G.dallorto

L4a1

4, 4 ++ ++ | 2, 3 ++ −+ | 2, 3 ++ +− | 0, 2 ++ −−
2, 3 +− ++ | 0, 2 +− −+ | 0, 2 +− +− | −2, 1 +− −−
2, 3 −+ ++ | 0, 2 −+ −+ | 0, 2 −+ +− | −2, 1 −+ −−
0, 2 −− ++ | −2, 1 −− −+ | −2, 1 −− +− | −4, 2 −− −−

A **knot invariant** is a quantity which is invariant under deformation of a knot or link, without cutting, in three dimensions. Equivalently, we define the invariance as under application of the three **Reidemeister Moves** to a 2D knot diagram representing the three-dimensional embedding schematically, in terms of over- and under-crossings. Our example concerns a 2-component link, the so-called 'Solomon's Seal' knot (above) labelled, as an alternating 4-crossing link, L4a1. Each component is the 'unknot' whose diagram is a simple closed plane curve: ◯. Now the Kauffman bracket for a collection of k separated unknots ◯···◯ may be specified as $\langle ◯···◯\rangle = (-x^2 - x^{-2})^{k-1}$. But L4a1's unknots are intertwined; this is resolved by a 'smoothing' operation:

> while ascending an over-crossing, ⊣⊢, we switch to the under-crossing, either to the left, ⌐∟, a '+1 smoothing' or the right, ⌐, a '−1 smoothing' (the definition is rotated appropriately for non-vertical over-crossings).

There is a corresponding definition of ⟨K⟩ as a recursive rule:
$$\langle\text{⊣⊢}\rangle = x\langle\text{⌐∟}\rangle + x^{-1}\langle\text{⌐}\rangle.$$

A complete smoothing of an n-crossing knot K is thus a choice of one of 2^n sequences of +'s and −'s. Let S be the collection of all such sequences. Each results in, say k_i, separate unknots, and has a 'positivity', say p_i, obtained by summing the ±1 values of its smoothings. Then the full definition of ⟨K⟩ is:
$$\langle K\rangle = \sum_S x^{p_i}(-x^2 - x^{-2})^{k_i - 1}.$$

Above right, the 2^4 pairs (p_i, k_i) give us $\langle L4a1\rangle = x^4(-x^2 - x^{-2})^3 + 4x^2(x^2 - x^{-2})^2 + 6x^0(x^2 - x^{-2})^1 + 4x^{-2}(x^2 - x^{-2})^0 + x^{-4}(x^2 - x^{-2})^1 = -x^{10} + x^6 - x^2 - x^{-6}$.

Finally we factor in the **writhe**. This requires an arbitrary orientation of each component of a link K. Choose the smoothing sign at each crossing consistently with the orientation. Then the writhe $w(K)$ is the positivity of the complete smoothing. If the components of our example are oriented counterclockwise then the smoothing signs are $-1 -1 -1 -1$, as in the final smoothing above right, giving $w(K) = -4$. And we derive $J(L4a1) = (-x)^{3\times-4}\langle L4a1\rangle = -x^{-2} + x^{-6} - x^{-10} - x^{-18}$.

Vaughan Jones' polynomial (1984), with Louis Kauffman's bracket 'operator' reformulation (1987), gave knot theory an important new tool and created deep links with other areas of mathematics and with theoretical physics.

Web link: sites.google.com/view/alanchang/papers (see 'Expository'). Crosscheck $J(L4a1)$ at katlas.org/wiki/The_Thistlethwaite_Link_Table, with $x = q^{1/4}$.
Further reading: *The Knot Book* by Colin Adams, AMS, 2004, chapter 6.