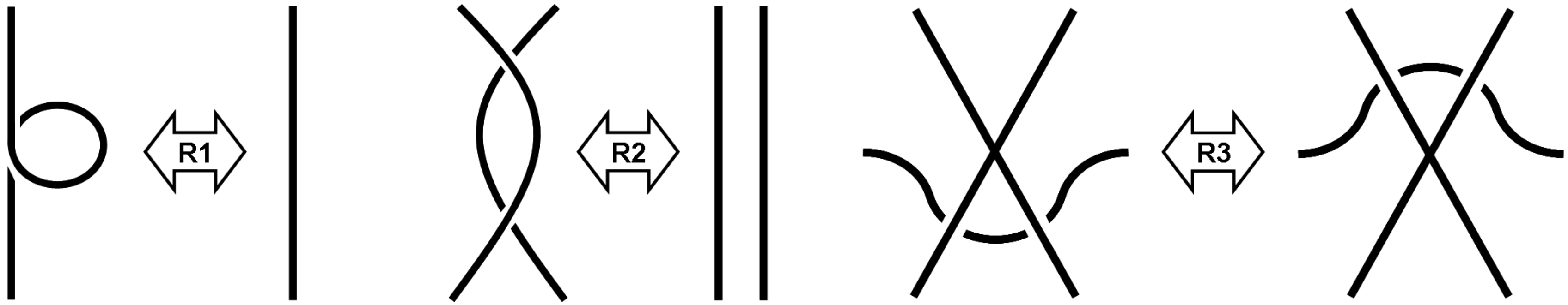
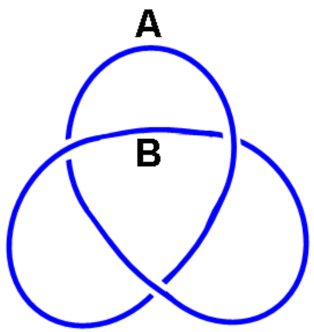
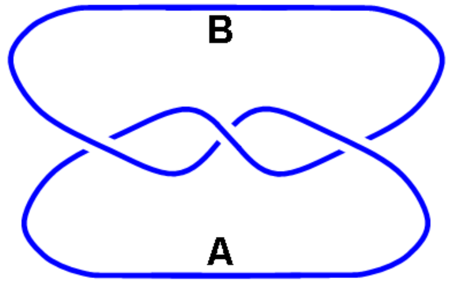
THEOREM OF THE DAY

Reidemeister's Theorem *Two knots are topologically equivalent if and only if their projections may be deformed into each other by a sequence of the three moves shown below.*



It is not usually at all obvious how to determine whether one knot (an embedding in three dimensions of a closed loop) is equivalent to another (that is, transformable without cutting or intersecting the loops). For example, the two knots shown here:





are both projections (into two dimensions) of the so-called *trefoil* knot. Imagining things in three dimensions you may convince yourself that, by pulling part A, on the left, down over the whole knot, and then pulling part B out into a large loop, you have, without changing the knot, arrived at the right-hand projection. Reidemeister's Theorem guarantees that you can accomplish the transformation using some sequence of just the three *Reidemeister Moves:* R1, R2 and R3. This may not, in general, implement your 3D transformation! Thus, one sequence of moves for the above knots begins by creating a loop (move R1) at part A of the trefoil and then moving (using R3) the double over-pass thus created across the under-pass to its left: see the diagram on the right.

The complete transformation is the sequence R1, R3, R3, R1, with some 'cosmetic' (not affecting the crossings) stretching and displacing of arcs. Try it yourself (link to solution below).

This was a pivotal theorem which helped make knots an object of rigorous mathematical study. It is one of those theorems proved simultaneously and independently in two places: in this case in 1926 by Kurt Reidemeister in Königsberg and by J.W. Alexander and G.B Briggs in Princeton. Today, knot theory has important overlaps with mathematical physics, as well as many branches of pure mathematics beyond topology.

Web link: tmattman.yourweb.csuchico.eud/NSF/Lecturenotes.pdf (1MB): on p. 20 the above trefoil equivalence is very nicely demonstrated; also (p. 26), a sketch-proof of the non-trivial 'only if' part of the theorem, with a reference for locating a full proof.

Further reading: *A Topological Aperitif, 2nd edition* by Stephen A. Huggett and David Jordan, Springer, London, 2009.