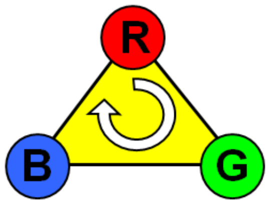
THEOREM OF THE DAY

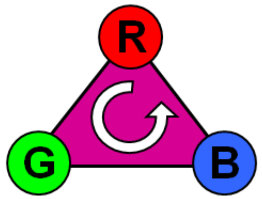
Sperner's Lemma *Suppose the interior of a triangle is triangulated (that is, divided up internally into small triangles). The vertices of the triangle are coloured red, green and blue, respectively. All other vertices, where lines meet inside or around the outside edges of the triangle, are also coloured red, green or blue with the restriction that no edge of the main triangle contains all three colours. Let T_C be the number of small triangles whose vertices are coloured red, green and blue in* clockwise *order; and let T_A be the number of small triangles whose vertices are coloured red, green and blue in* anticlockwise *order. Then*

$$|T_C - T_A| = 1.$$

In particular, the total number of small red-green-blue triangles must be odd, and is certainly never zero.



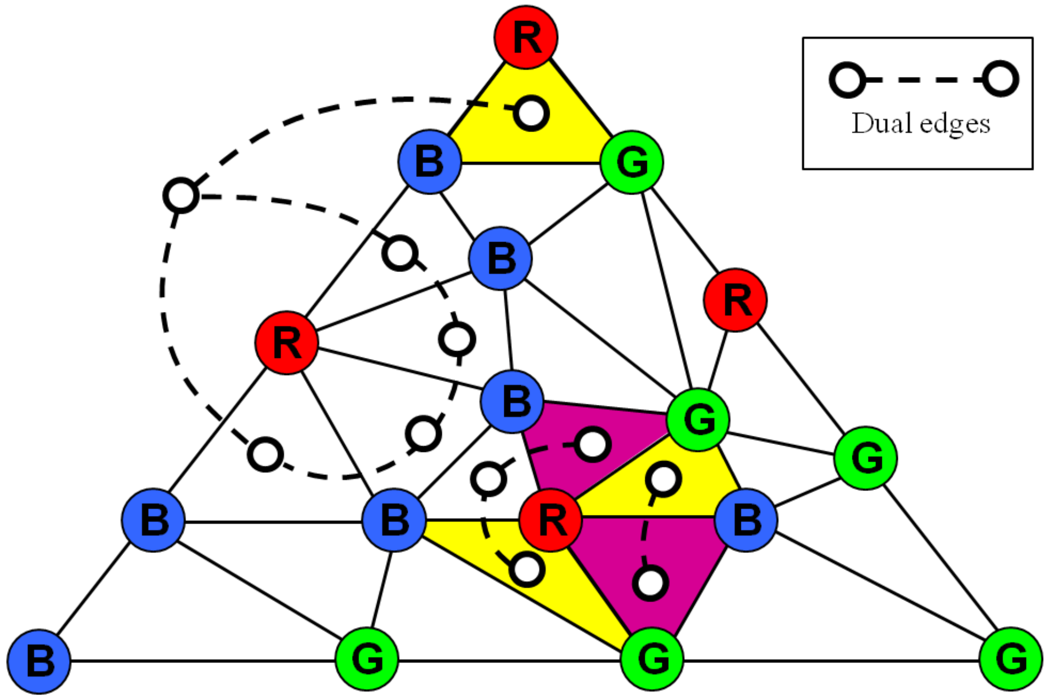
Counted by T_C



Counted by T_A

The triangulation shown on the right has been given a so-called *Sperner colouring*. The small triangles which have vertices of all three colours have been colour-coded as shown above and you can check that $T_C = 3$ and $T_A = 2$.

In addition, we have added a 'dual' edge joining dual vertices across every red-blue triangulation edge. The only dual vertices having odd degree (odd number of incident dual edges) are the outside vertex and those inside small red-green-blue triangles. The Handshaking Lemma says that odd degree vertices are even in number. So the number of small red-green-blue triangles must be odd.



In two dimensions this lemma, due to Emanuel Sperner (1928), is a charming curiosity; but it extends to arbitrary dimensions (tetrahedra replacing triangles for 3D, etc) and is then found to be equivalent to the powerful and important Brouwer fixed point theorem.

Web link: jonathan-huang.org/research/old/sperner.pdf

Further reading: *A Combinatorial Introduction to Topology* by Michael Henle, Dover Publications, 1994