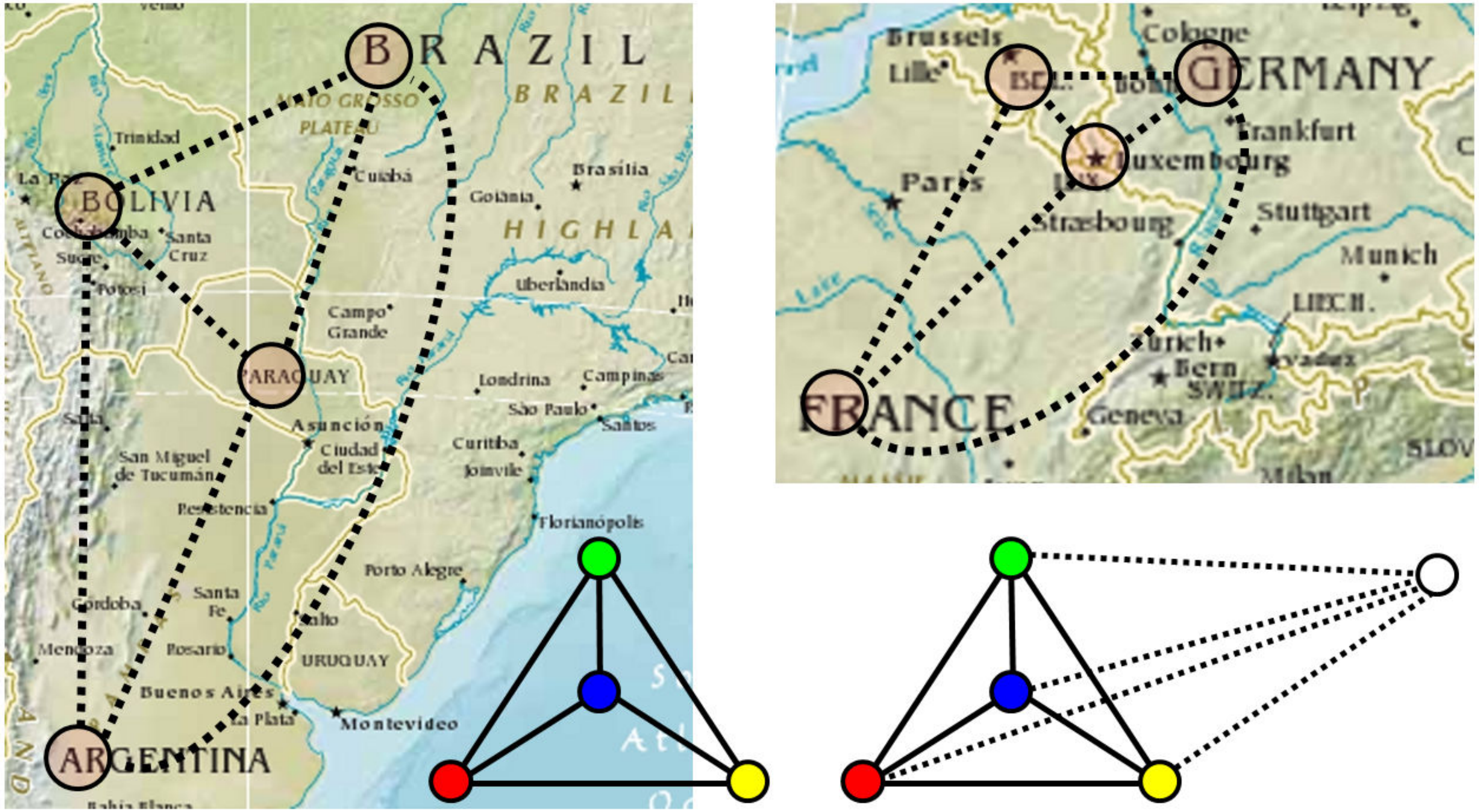
The Four-Colour Theorem *Any planar graph may be properly coloured using no more than four colours.*



Question: can we colour the countries on a map of South America or of Europe using just four colours so that no countries sharing a common border are coloured the same? A classic piece of mathematical modelling disregards all non-essential geography, replacing countries by *vertices* and the common border relation by *edges* to produce a *graph*. A colouring of the vertices is *proper* if no two vertices of the same colour are joined by an edge: this captures the map colouring requirement. A graph is *planar* if it may be drawn without edges intersecting (except at their end-vertices) and our map model is clearly endowed with this property. The maps on the right both contain four mutually bordering countries and this corresponds to a four-vertex *complete graph* necessitating four colours, as illustrated centre-right. A further country bordering all four simultaneously would then require a fifth colour; but this produces a *nonplanar* graph (far right): which we agree cannot happen in standard cartography. The challenge is to show that there can be no pathological planar configuration which defies four-colourability.

The theorem arose first, in the form of the map-colouring question, in 1852, proposed by Francis Guthrie and popularised by Augustus De Morgan. A defective proof (1879) by Alfred Bray Kempe, perhaps the most famous mathematical error in history, nevertheless contained the essence of the eventual proof, almost one hundred years later. This consists in identifying an **unavoidable** set S of configurations which are **reducible**, i.e. produce a smaller graph for which a 4-colouring implies a 4-colouring of the original: this refutes existence of a smallest counterexample to the theorem. A general theory of reducibility was established by George David Birkhoff in the early 1900s; the problem of certifying unavoidability for (necessarily) very large sets of configurations was made accessible to computer search by the 'discharging method' of Heinrich Heesch (1969) and a proof was announced in 1976 by Ken Appel, Wolfgang Haken and John Koch, the latter credited for the computer checking required by the proof. The reliance on a computer made the proof controversial; a shorter, more elegant proof, based on the same ideas and still requiring a computer, was published 20 years later by Robertson, Sanders, Seymour and Thomas. The theorem is now considered established beyond doubt although it is fair to say that a deep understanding of graph colouring still eludes us.

Web link: www.math.gatech.edu/~thomas/FC/fourcolor.html. The map images are from www.lib.utexas.edu/maps/.
Further reading: *Four Colors Suffice: How the Map Problem Was Solved* by Robin Wilson, Princeton University Press, revised colour edition, 2013.